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Gyanabha
Vol 18 1988
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ISSN 0304 -9892

Jñānābha ज्ञानाभ

VOLUME 18

1988

Published by :

The Vijnana Parishad ef India

DAYANAND VEDIC POSTGRADUATE COLLEGE

Bundelkhand University

ORAI, U. P., INDIA

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Jñānābha, Vol. 18, 1988

SOME MULTIPLIERS FOR THE | C, 1 | SUMMABILITY OF LAPLACE SERIES

By

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(Received : June 17, 1986 ; Revised : November 14, 1987)

ABSTRACT

In the present paper a new theorem on the absolute $(C,1)$ summability of Laplace series has been proved.

1. INTRODUCTION

Let $f(\theta,\phi)$ be a function defined for the range $0\leqslant\theta\leqslant\pi$, $0\leqslant\phi\leqslant 2\pi$ and integrable (L) over the surface of the sphere S. The Laplace series associated with this function is

$$f(\theta,\phi) \sim \frac{1}{2\pi} \sum_{n=0}^{\infty} (n+\tfrac{1}{2}) \iint_S f(\theta',\phi') P_n(\cos w)\, dS', \tag{1.1}$$

where

$$\cos w = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

$$dS' = \sin\theta'\, d\theta'\, d\phi' .$$

The Legendre polynomials $P_n(x)$ are defined by the relation

$$(1-2\mu x+\mu^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)\,\mu^n . \tag{1.2}$$

Following Kogbetliantz [3] the general value of $f(\theta,\phi)$ is

$$\phi(w) = \frac{1}{2\pi\sin w} \int_{C_w} f(\theta',\phi')\, ds' , \tag{1.3}$$

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where the curvilinear integral is taken along the small circle C_w, whose centre is (θ,ϕ) on the sphere and whose curvilinear radius is w. In view of the relation (1.3) the series (1.1) reduces to the form

$$\sum_{n=0}^{\infty} (n+\tfrac{1}{2}) \int_{0}^{\pi} \phi(w)\, P_n(\cos w)\, \sin w\, dw \equiv \sum_{n=0}^{\infty} A_n . \qquad (1.4)$$

We write

$$\phi(t) = \int_{t}^{\delta} w^{-1,2}\, |\phi(w)|\, dw, \qquad 0<\delta\leqslant\pi$$

$$\beta_n = \prod_{\mu=1}^{k-1} (log^{\mu} n)\, (log^{k} n)^{1+\epsilon}, \quad \epsilon>0$$

where

$$log^{k} n = log\,(log^{k-1} n), \quad ., log^{2} n = log\, log\, n,$$

and prove the following

THEOREM. *If* $\{\lambda_n\}$ *is a convex sequence such that* $\sum_{n=1}^{\infty} \lambda_n/n<\infty$, *and if*

$$\phi(t) = O\{t^{\frac{1}{2}} (log^{k}(1/t))^{\eta}\} \text{ as } t\to+0, \qquad (1.5)$$

then the series

$$\Sigma\lambda_n A_n/\beta_n$$

is summable $|C,1|$ *for* $0<\eta<\epsilon$.

2. RESULTS REQUIRED

The following lemmas are required in our investigation :

Lemma 1.

$$|P_n(\cos\theta)| = O(1) \text{ for } 0\leqslant\theta\leqslant\pi .$$

Lemma 2 [1] .

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$$- \frac{\pi}{2} \left[\frac{d}{dx} \{ P_{n+1}(x) + P_n(x) \}_{x=\cos w} \right] \sin w$$

$$= R \Big\{ (n+1) e^{i(n+1)w + \pi/4} [\cot \tfrac{1}{2}w]^{\frac{1}{2}} \int_0^1 (1-u)^n u^{-\frac{1}{2}} g(w,u)\,du$$

$$- (n+1) e^{i(n+3/2)w + \pi/4} (2 \sin w)^{-\frac{1}{2}} \int_0^1 (1-u)^n u^{-\frac{1}{2}} g(w,u)\,du$$

$$+ e^{i(n+3/2)w + \pi/4} (2 \sin w)^{-\frac{1}{2}} \int_0^1 (1-u)^{n+1} u^{-\frac{1}{2}} g(w,u)\,du$$

$$+ \tfrac{1}{2} e^{i(n+2)w + 2\pi/4} (\sin \tfrac{1}{2}w)^{-1} (2 \sin w)^{-\frac{1}{2}} \int_0^1 (1-u)^{n+1}$$

$$u^{-\frac{1}{2}} [g(w,u)]^3\,du$$

$$- e^{i(n+5/2)w + 3\pi/4} (2 \sin w)^{-3/2} \int_0^1 (1-u)^{n+1} u^{\frac{1}{2}} [g(w,u)]^3\,du \Big\},$$

where

$$g(w,u) = \{ 1 - ue^{iw} (2i \sin w)^{-1} \}^{-\frac{1}{2}} \leqslant \sqrt{2}.$$

Now let this be equal to $J_1+J_2+J_3+J_4+J_5$, say, where

$$J_1 = A \frac{(n+1)\Gamma(n+1)}{\Gamma(n+3/2)} (\cot \tfrac{1}{2}w)^{\frac{1}{2}} e^{i(n+1)w}$$

$$J_2 = A \frac{(n+1)\Gamma(n+1)}{\Gamma(n+5/2)} (\sin w)^{-\frac{1}{2}} e^{i(n+3/2)w}$$

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$$J_3 = A \frac{\Gamma(n+2)}{\Gamma(n+5/2)} (\sin w)^{-\frac{1}{2}} e^{i(n+3/2)w}$$

$$J_4 = A \frac{\Gamma(n+2)}{\Gamma(n+5/2)} (\cos \tfrac{1}{2}w)^{-\frac{1}{2}} (\sin \tfrac{1}{2}w)^{-3/2} e^{i(n+5/2)w}$$

$$J_5 = A \frac{\Gamma(n+2)}{\Gamma(n+5/2)} (\sin w)^{-5/2} e^{i(n+5/2)w},$$

and A is a constant, not necessary the same at each occurrence.

3. PROOF OF THE THEOREM

Let τ_n be the *nth* Cesàro mean of first order of the sequence $\{ n\lambda_n A_n/\beta_n \}$. It is sufficient to prove that

$$\sum_{n=2}^{\infty} |\tau_n| / n < \infty .$$

$$\tau_n = \frac{1}{n+1} \int_0^{\pi} \phi(w) \left\{ \sum_{\nu=2}^{n} (\nu+\tfrac{1}{2}) P_\nu(\cos w) \sin w \cdot \nu\lambda_\nu \beta_\nu^{-1} \right\} dw$$

$$= \int_0^{1/n} + \int_{1/n}^{\pi-(1/n)} + \int_{\pi-(1/n)}^{\pi}$$

$$= I_1 + I_2 + I_3, \text{ say.}$$

$$|I_1| = 0(n^{-1}) \int_0^{1/n} w |\phi(w)| dw \cdot \sum_{\nu=2}^{n} \nu(\nu+\tfrac{1}{2}) \lambda_\nu \beta_\nu^{-1}$$

Now

$$\int_0^t |\phi(w)| dw = \int_0^t w^{\frac{1}{2}} \phi'(w) dw$$

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$$= \left[w^{\frac{1}{2}} \phi(w) \right]_0^t - \frac{1}{2} \int_0^t w^{-\frac{1}{2}} \phi(w)\, dw$$

$$= O\left[w (log^k (1/w))^n \right]_0^t - O\left\{ \int_0^t \left[w^{\alpha} (log^k (1/w))^n \right] w^{-\alpha}\, dw \right\}, \quad 0 < \alpha < 1$$

$$= O \{ t (log^k (1/t))^n \} .$$

Hence

$$| I_1 | = O (n \beta_n^{-1}) \int_0^{1/n} | \phi(w) |\, dw \sum_{\nu=2}^{n} \frac{\lambda_\nu}{\nu}$$

$$= O \{ (log^k n)^n / \beta_n \} . \qquad (3.1)$$

By Abel's transformation,

$$I_2 = \frac{1}{n+1} \int_{1/n}^{\pi-(1/n)} \phi(w) \left\{ \sum_{\nu=0}^{n-1} \sum_{r=0}^{\nu} (r+\tfrac{1}{2}) P_r (cos\, w) \sin w \hat{a}.$$

$$\triangle (\nu \lambda_\nu \beta_\nu^{-1}) + \sum_{\nu=0}^{n} (\nu + \tfrac{1}{2}) P_\nu (cos\, w) \sin w \,.\, n \lambda_n \beta_n^{-1} \Big\} dw$$

$$= \frac{1}{n+1} \int_{1/n}^{\pi-(1/n)} \phi(w) \left\{ \sum_{\nu=0}^{n-1} -\frac{d}{dx} \left[P_{\nu+1}(x) + P_\nu(x) \right]_{x=cos\, w} \right.$$

$$\sin w \,.\, \triangle (\nu \lambda_\nu \beta_\nu^{-1})$$

$$\left. + -\frac{d}{dx} \left[P_{n+1}(x) + P_n(x) \right]_{x=cos\, w} \sin w . n \lambda_n \alpha_n^{-1} \right\} dw$$

$$= I_{2,1} + I_{2,2} \ , \ say \ .$$

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$$I_{2,1} = \frac{1}{n+1} \int_{1/n}^{\pi-(1/n)} \phi(w) \left\{ \sum_{\nu=0}^{n-1} [J_1 + J_2 + J_3 + J_4 + J_5] \triangle(\nu \lambda_\nu \beta_\nu^{-1}) \right\} dw$$

$$= I_{2,1,1} + I_{2,1,2} + I_{2,1,3} + I_{2,1,4} + I_{2,1,5}, \text{ say}.$$

As the other cases are similar, we consider only $I_{2,1,1}$ and $I_{2,1,4}$.

Since

$$\triangle(\nu \lambda_\nu \beta_\nu^{-1}) = \nu \lambda_\nu \triangle \beta_\nu^{-1} + \nu \beta_{\nu+1}^{-1} \triangle \lambda_{\nu-\lambda_{\nu+1}} \beta_{\nu+1}^{-1}$$

$$= O(\lambda_\nu \beta_\nu^{-1}) + O(\nu \beta_\nu^{-1} \triangle \lambda_\nu)$$

therefore

$$| I_{2,1,1} | = O(n^{-1}) \int_{1/n}^{\pi-(1/n)} w^{-\frac{1}{2}} | \phi(w) | dw \left\{ \sum_{\nu=0}^{n-1} \nu^{\frac{1}{2}} \left[O(\lambda_\nu \beta_\nu^{-1}) + O(\nu \beta_\nu^{-1} \triangle \lambda_\nu) \right] \right\}$$

$$= O\{n^{-3/2} (\log^k n)^\eta\} \left\{ O(n^{3/2} \beta_n^{-1}) \sum_{\nu=0}^{n-1} \frac{\lambda_\nu}{\nu} + O(n^{3/2} \beta_n^{-1}) \sum_{\nu=0}^{n-1} \triangle \lambda_\nu \right\}$$

$$= O\{ (\log^k n)^\eta / \beta_n \}$$

and

$$| I_{2,1,4} | = O(n^{-1}) \int_{1/n}^{\pi-(1/n)} w^{-3/2} | \phi(w) | dw \left\{ \sum_{\nu=0}^{n-1} \nu^{-\frac{1}{2}} \left[O(\lambda_\nu \beta_\nu^{-1}) + O(\nu \beta_\nu^{-1} \triangle \lambda_\nu) \right] \right\}$$

$$= O\left\{ n^{-\frac{1}{2}} (\log^k n)^\eta \right\} \cdot \left\{ O(n^{\frac{1}{2}} \beta_n^{-1}) \sum_{\nu=0}^{n-1} \frac{\lambda_\nu}{\nu} + O(n^{\frac{1}{2}} \beta_n^{-1}) \sum_{\nu=0}^{n-1} \triangle \lambda_\nu \right\}$$

$$= O\{ (\log^k n)^\eta / \beta_n \}.$$

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Hence

$$| I_{2,1} | = O \{ (log^k n)^\tau / \beta_n \} .$$

$$I_{2,2} = \frac{n\lambda_n \beta_n^{-1}}{n+1} \int_{1/n}^{\pi-(1/n)} \phi(w) [J_1 + J_2 + J_3 + J_4 + J_5] \, dw$$

$$= I_{2,2,1} + I_{2,2,2} + I_{2,2,3} + I_{2,2,4} + I_{2,2,5} , \ say .$$

$$| I_{2,2,1} | = O(n^{1/2} \lambda_n \beta_n^{-1}) \int_{1/n}^{\pi-(1/n)} w^{-\frac{1}{2}} | \phi(w) | \, dw = O \left\{ \lambda_n (log^k n)^\tau / \beta_n \right\} .$$

$$| I_{2,2,4} | = O(n^{-\frac{1}{2}} \lambda_n \beta_n^{-1}) \int_{1/n}^{\pi-(1/n)} w^{-3/2} | \phi(w) | \, dw = O \left\{ \lambda_n (log^k n)/\tau \beta_n \right\} .$$

Therefore

$$| I_{2,2} | = O \{ \lambda_n (log^k n)^\tau / \beta_n \} .$$

and hence

$$| I_2 | = O \{ (log^k n)^\tau / \beta_n \} . \qquad (3.2)$$

Lastly, we consider

$$| I_3 | = O (n^{-1}) \int_{\pi-(1/n)}^{\pi} | \phi(w) | \, sin \, w \, dw \sum_{\nu=2}^{n} \nu (\nu + 1) \lambda_\nu \beta_\nu^{-1}$$

$$= O (n^{-1}) \int_0^{1/n} w \, dw \, . \, O (n^3 \beta_n^{-1}) \sum_{\nu=2}^{n} \frac{\lambda_\nu}{\nu}$$

$$= O (\beta_n^{-1}) . \qquad (3.3)$$

Combining (3.1), (3.2) and (3.3) , we obtain

$$| \tau_n | = O \{ (log^k n)^\tau / \beta_n \} .$$

Therefore

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$$\Sigma \mid \tau_n \mid /n = O\left\{\sum \frac{1}{n\left[\prod_{\mu=1}^{k-1} \log^{\mu} n\right](\log^{k} n)^{1+\epsilon-\eta}}\right\}$$

$$= O(1) .$$

This completes the proof of the theorem .

ACKNOWLEDGEMENTS

I am very much thankful to Professor H. M. Srivastava for his kind help and advice during the preparation of this paper .

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Jñānābha, Vol. 18, 1988

IPER - DUAL SYSTEMS AND HILBERT TRIPLETS

By

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(*Received : November 15, 1987*)

ABSTRACT

We introduce the concept of iper-dual systems . Beginning from the iper-dual system $[H^*,S>$, we construct the Hilbert triplet $S\subseteq H\subseteq S'$, where S is a pre-Hilbert space dense in the (real or complex) Hilbert space H and S' is a space of linear functionals on S wider that H. The couple $[S',S>$ so obtained is an iperdual system . Once having introduced the order-bounded nets, we prove that if every weak-convergent net is order-bounded, then $S=H$. Finally we study four examples of special triplet $S\subseteq H\subseteq S'$.

0. INTRODUCTION

In Section 1 we introduce the concept of iper-dual systems, as a class of couples of linear spaces placed between the class of dual systems [1], and the class of Hilbert spaces .

We construct a space of linear functionals X' by means of the X-weak Cauchy sequences so that $X\subseteq\hat{X}\subseteq X'$, where $[\hat{X}, X>$ is a iper-dual system .

In Section 2 we consider the iper-dual system $[H^*, S>$, where S is a pre-Hilbert space, H is the Hilbert space completion of S and H^* is the dual of H. We construct the triplet $S\subseteq H\subseteq S'$ from $[H^*,S>$, in such a way that $[S', S>$ is an iper-dual system, too. The main result obtained in this section is that if every weak convergent net is order-bounded, then $S=H$.

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Finally in Section 3 we study four examples of triplet $S \subseteq H \subseteq S'$, In the exemple 1, H is a separable, complex Hilbert space with a fixed orthonormal basis $\{u_n\}$, $S := Sp(\{u_n\})$ is the linear manifold spanned by $\{u_n\}$ and S' is constructed as descripted in Section 1. In this context we show that S' is identifiable with C^N, the space of all complex sequences.

In the example 2, $H = L_2(R^n)$ and S is the space $S(R^n)$ of functions of rapid decrease in R^n. In this case S' coincides with the space of temperated distributions.

In the example 3, $H = l_2$ and $S = \left\{ \{a_n\} : \sum_{n=1}^{\infty} |a_n|^2 (n+1)^m < \infty \ \forall m \in N \right\}$.

We show that $S' = \{ \{b_n\} : \exists c \in R \text{ and } \exists m \in N \text{ such that } |b_n| \leqslant c(1+n)^m \ \forall n \in N \}$, who coincides with the dual of S with respect to the topology of conutably normed space that makes S isomorphic to $S(R)$.

In the example 4, H is the space of Hilbert- Schmidt operators on a separable Hilbert space $\mathbf{H}$ and S is the space of finite rank operators on $\mathbf{H}$. If we denote with $K(\mathbf{H})$ and $B(\mathbf{H})$ the space of compact operators and the space of bounded operators on $\mathbf{H}$ resp ectively, we show that $K(H) \subseteq S' \subseteq B(H)$.

It is useful to observe that in all previous examples, S is not complete in the topology induced by inner product. The fact that S is not complete is the necessary and sufficient condition in order that we obtain an *actual* enlargment S' of H^*.

1. IPER-DUAL SYSTEMS AND WEAK CAUCHY SEQUENCES

Let $\hat{X}, X$ be two linear spaces. We suppose that :

(*IDS*-1) X is a linear manifold of $\hat{X}$.

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(*IDS*-2) There is a functional $[,>:\hat{X}\times X\to C$ such that

(a) the restriction of $[,>$ to $X\times X$ is an inner product on X denoted by $<1>$.

(b) $[\hat{x}, . >$ is a linear functional for every $\hat{x}\epsilon\hat{X}$

(c) $[.,x>$ is an antilinear functional for every $x\epsilon X$.

(d) $[\hat{x},x>=0\ \forall x\epsilon X\Rightarrow\hat{x}=0$.

Set $\bar{R}=\mathbb{R}\cup\{+\omega\}$. We define the function (called infi-norm) $\|\ \|_\wedge : \hat{X}\to\bar{R}$ in the following manner :

$$\|\hat{x}\|_\wedge = sup\ \{\ |\ [\ \hat{x},y>\ |\ :\ y\epsilon X,\ \|y\|=1\}$$

where, of course, $\|y\|^2=<y|y>$. Then we have

(N-1) $\|\hat{x}\|_\wedge=0$ *iff* $\hat{x}=\underline{0}$

(N-2) $\|\lambda\hat{x}\|_\wedge = |\lambda|\ \|\hat{x}\|_\wedge\ \ \forall\lambda\epsilon C$

(N-3) $\|\hat{x}+\hat{y}\|_\wedge \leqslant \|\hat{x}\|_\wedge+\|\hat{y}\|_\wedge$

We remark that the generalized Schwartz inequality holds :

$$|\ [\ \hat{x},\ x>\ |\ \leqslant\|\hat{x}\|_\wedge\ \|x\|\quad\forall\hat{x}\epsilon\hat{X}\ ,\ \forall x\epsilon X\ .$$

(of course, $\|x\|_\wedge=\|x\|\ \forall x\epsilon X$) .

DEFINITION.

We say that $[\hat{X}, X>$ is an iper-dual system if (IDS-1), (IDS-2) hold and moreover

(IDS-3) $\bar{X}\subseteq\hat{X}$, where $\bar{X}$ is the Hilbert space completion of

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X with respect to the inner prduct $<1>$.

Let $[\hat{X}, X>$ be an iper-dual system. We say that a sequence $\{\hat{x}_n\} \subseteq \hat{X}$ is X-weak Cauchy *iff* $\exists \lim_n [\hat{x}_n, y> \ \forall y \in X$.

We say $\{\hat{x}_n\}$ is X-weak convergent to $\hat{x} \in \hat{X}$ *iff* $\lim_n [\hat{x}_n, y> = [\hat{x}, y> \ \forall y \in X$.

We see immediately that the criterion of weak convergence defined on X is compatible with its linear structure [2] .

Furthermore, we observe that there exist X-weak convergent sequences for which the corresponding sequences of infi-norms do not converge to the infi-norm of limit .

Let $L_w(\hat{X}, X)$ denote the linear space of the X-weak Cauchy sequences in $\hat{X}$. Every X-weak Cauchy sequence $\{\hat{x}_n\}$ defines a linear functional $t : X \to C$ by the law $<t, x> := \lim_n [\hat{x}_n, x> \ \forall x \in X$.

The set of all linear functionals generated by some X-weak Cauchy sequence is denoted by X' . In the same way, generalizing the notion of X-weak Cauchy sequence, we shall say that a sequeuce $\{t_n\}$ in X' is a X-weak Cauchy sequence *iff* $\exists \lim_n <t_n, x> \forall x \in X$.

Any X-weak convergent sequence in X' is a X-weak Cauchy sequence too . Now let $\{t_n\}$ be a X-weak Cauchy sequence in X', then we can define the linear functional $l : X \to C$ *as* $<l, x> = \lim_n <t_n, x> \ \forall x \in X$ and we write $l = w - \lim t_n$.

In general l is not an element of X since it is not assured that there exists a sequence $\{\hat{x}_n\}$ in $\hat{X}$ which converges X-weak to l. In the next proposition we shall give a sufficient condition ensuring that $l \in X$:,

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PROPOSITION 1. *Let* $\{t_n\}$ be a X-weak Cauchy sequence in X'.

If any t_n is generated by a sequence $\{\hat{x}_j^{(n)}\}_{j\in N}$ in $\hat{X}$ which is uniformly weak convergent to t_n in the sense that for every $n\in N$ and every $y\in X$ there exists an integer $j=j(n)$ depending on n but not on y, such that

$$|<t_n, y>-[\hat{x}_{j(n)}^{(n)}, y>|\leqslant\frac{1}{n}\quad\forall y\in X$$

then $l=w\text{-}lim\ t_n$ is an element of X'.

PROOF. Considering the sequence $\{\hat{x}_{j(n)}^{(n)}\}$ in $\hat{X}$ we get that it is a X-weak Cauchy sequence; indeed, from

$$|[\hat{x}_{j(n)}^{(n)}, y>-[\hat{x}_{j(m)}^{(m)}, y>|\leqslant|<t_n, y>-[\hat{x}_{j(n)}^{(n)}, y>|+$$

$$|<t_n, y>-<t_m, y>|+|<t_m, y>-[\hat{x}_{j(m)}^{(m)}, y>|$$

it follows that $\exists\ lim\ [\hat{x}_{j(n)}^{(n)}, y>\ \forall y\in X$.

On the other hand, the relation

$$|<l, y>-[\hat{x}_{j(n)}^{(n)}, y>|\leqslant|<l, y>-<t_n, y>|+|<t_n, y>-[\hat{x}_{j(n)}^{(n)}, y>|$$

under our hypotheses gives the result $l\in X'$.

On $L_w(\hat{X}, X)$ we can introduce the following equivalence relation, called the X-weak equiconvergence relation, defined as follows :

$$\{\hat{x}_n'\}\sim_w\{\hat{x}_n''\}\ \textit{iff}\ \lim_n[\hat{x}_n', y>=\lim_n[\hat{x}_n'', y>\ \forall y\in X.$$

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It is trivial to see that X' and $L_w(\hat{X}, X)/\sim_w$ are linearly isomorphic. Set $t \equiv [\{\hat{x}_n\}]\sim_w$. We suppose that $\|\hat{x}_n\|_\wedge < M$. If the sequence $\{y_k\} \subseteq X$ is norm-convergent to zero, then

$$0 \leqslant |<t, y_k>| = \lim_n |[\hat{x}_n, y>| \leqslant \sup_n \|\hat{x}_n\|_\wedge \cdot \|y_k\| \leqslant M\|y_k\|$$

so that $\lim_k <t, y_k> = 0$. Thus t is sequentially continuous (and therefore continuous) with respect to the topology on X induced by the inner product. Hence there exists $x_0 \in \bar{X} \equiv \bar{X}^*$ such that $t = x_0$, where $\bar{X}^*$ denotes the dual space of $\bar{X}$ with respect to the topology induced by the inner product. Therefore we have proved that :

PROPOSITION 2.

If we denote by $L^b_w(\hat{X}, X)$ the linear subspace of $L_w(\hat{X}, X)$ of bounded sequences in infl-norm, then $\bar{X}$ and $L^b_w(\hat{X}, X)/\sim_w$ are linearly isomorphic. In particular, it follows that

$$\|x\|_\wedge < \infty \; \forall x \in \bar{X} \text{ and } \|\hat{x}\|_\wedge = +\infty \; \forall \hat{x} \in \hat{X} \setminus \bar{X}$$

2. HILBERT TRIPLETS AND WEAK CONVERGENT NETS

We shall construct now an iper-dual system by means of an other iper-dual system. In this Section, we denote by S a pre-Hilbert space with inner product $<1>$. $(S, T_{<1>}(S))$ denotes the locally convex topological vector spoce in which $T_{<1>}(S)$ is the topology induced by the inner product. S^* denotes the dual space of $(S, T_{<1>}(S))$ and H the Hilbert space completion of S. We denote by $D(S)$ the linear space antiisomorphic to S of all linear functionals $<x| : S \to C$, with $x \in S$, defined by $<x| : y \to <x|y> \; \forall y \in S$. We recall that $S = H$ iff $D(S) = S^*$. $[H^*, S>$ is an iper-dual system. We construct S' in the same manner as in the previous Section :

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$$S' = \{t : S \to C : \exists \{x_n\} \in L_w(H^*, S), t = w\text{-}\lim_n \langle x_n | \} .$$

We have that $S' \equiv L_w(H^*, S)/\sim_w$ and we have obtained the triplet

$$S \leqslant H \equiv H^* \equiv S^* \subseteq S'$$

$[S', S>$ is an iper-dual system .

REMARK 1. It is easy to prove that the Hilbert space H^* is identifiable with $L_w^b(D(S), S)/\sim_w$ in which bounded S-weak Cauchy sequences *from* $D(S)$ are involved. We shall remember that a standard procedure for constructing the completion H^* of $D(S)$ is based on strong Cauchy sequences using the strong equiconvergence relation

$$\{x_n'\} \sim_s \{x_n''\} \text{ iff } \lim_n \|x_n' - x_n''\| = 0 .$$

The above quoted result could seem very surprising, since S-weak Cauchy sequences are "much more" than strong Cauchy sequences from $D(S)$. However, it must be remarked that the S-weak equiconvergence relation is weaker than the strong equiconvergence relation, so that S-weak equiconvergence classes are "larger" than strong equiconvergence classes .

REMARK 2. If $\{x_n\}$ is a sequence from H* which is norm-bounded, then the existence of the $\lim_n \langle x_n | y \rangle \ \forall y \in S$ implies the existence of the $\lim_n \langle x_n | t \rangle \ \forall t \in H$. However, the conditions

(i) $\{x_n\} \sim_w \{x_n'\}$

(ii) $\exists \lim_n \langle x_n, t \rangle \ \forall t \in H$

(iii) $\|x_n\| \leqslant M$

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do not imply either the norm boundedness of $\{x_n'\}$ or the existence of the $\lim_n \langle x_n|t\rangle \ \forall t \in H$.

We can endow the space S^* with the $\sigma(S^*, S)$-topology generated by the basis of neighbourhoods of zero

$$U(x_1, \ldots, x_n;\varepsilon) = \{l \in S^* : |\langle l, x_j\rangle| < \varepsilon \ \forall j = 1, \ldots, n\},$$

$$x_1, \ldots, x_n \in S, \ \varepsilon \in R_+$$

so in this way $(S^*, \sigma(S^*,S))$ is an Hausdorff locally convex space whose topological dual is S. We recall that $\sigma(S^*,S)$ is the weakest topology for S^* with respect to which all the element of S are continuous. Hence we have that $\sigma(S^*,S) \leqslant T_{\langle 1 \rangle}(S^*)$.

The most important properties of the Hilbert triplet $S \subseteq H \equiv H^* \subseteq S'$ in relation to the corresponding iper-dual system ore sumarized in the following statements :

(a) $(S, \sigma(S, S'))$ *is a nuclear space* ;

(b) *the canonical embedding* $i : (S,\sigma(S,S')) \to (H,\sigma(H, H^*))$ *is continuous;*

(c) H^* *is dense in* $(S',\sigma(S',S))$.

PROOF. (a) The topology $\sigma(S, S')$ for S is the projective topology with respect to the family $\{(C_\alpha, L_\alpha) : C_\alpha = C, L_\alpha S\}$. Since each $C_\alpha = C$ is nuclear, then the space $(S, \sigma(S, S'))$ is nuclear too [3] ,

(c) We shall make the identification $H \equiv H^*$. Let $t \in S'$. Then $t = w\text{-}\lim_n x_n$, $\{x_n\} \in L_w(H^*,S)$. We shall define $t_k := w\text{-}\lim_n \langle x^{(k)}|$ where $\{x_n^{(k)}\}$ is the sequence in $L_w^b(H^*,S)$ defined by $\{x_1, x_2, \ldots, x_k, x_k, x_k, \ldots\}$. It results that $\langle t_k|, y\rangle = \langle x_k|y\rangle$ and so

$$\lim_k \langle t_k, y\rangle = \langle t, y\rangle \ \forall y \in S.$$

For sake of completeness we give now briefly some definitions and results useful in the sequel .

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A partially ordered set I is said to be directed if, given $\alpha,\beta\in I$ there exists $\gamma\in I$ such that $\alpha\in\gamma$ and $\beta\leqslant\gamma$.

A generalized sequence or a net in a topological space T is a mappiag $f: I\to T$, where I is a directed set. As in the case of sequences, we write $f(\alpha)=x_\alpha$ and we represent the net by its image $\{x_\alpha\}$.

Given the net $\{x_\alpha\}$ in the topological vector space T, we say that $x\in T$ is the limit of $\{x_\alpha\}$ (and we write $x=lim\ x_\alpha$) or that x_α converges to x if for any neighbourhood U_x of x there exists $\alpha_0\in I$ such that $x_\alpha\in U_x\ \forall\alpha\geqslant\alpha_0$. It is well known that a function f from a topological space S to a topological space T is continuous iff for every convergent net $\{x_\alpha\}$ in S, with $x_\alpha\to x$, the net $f(x_\alpha)$ converges in T to $f(x)$. A net $\{x_\alpha\}$ in S is said to be order-bounded if there exists $\alpha_0\in I$ snch that $\{x_\alpha : \alpha\geqslant\alpha_0\}$ is norm-bounded .

The main result of this section is the following

THEOREM. (a) $S=H \Rightarrow L_w(H^*,S) = L_w^c(H^*, S)$

(b) If every net $\sigma(S, D(S))$ - convergent is order-bounded, then $S=H$.

PROOF. (a) It is an obvious consequence of the Uniform Boundedness Principle [4] .

(b) It is sufficient to prove that $D(S)=S^*$. Since the topological dual of $(S, \sigma(S, D(S))$ is $D(S)$, it is sufficient to show that every $l\in S^*$ is $\sigma(S, D(S))$ - continuous on S. For this, let $l\in S^*$ and let $\{x_\alpha\}$ be a net $\sigma(S, D(S))$ - convergent to x . Since $l\in S^*$, there exists $\{y_k\}\subseteq D(S)$ such that $\|L-y_k\|\underset{k\to\infty}{\to} 0$. Therefore

$|<l, x_\alpha-x>|=|<l\pm y_k, x_\alpha-x>|\leqslant \|l-y_k\|(\|x_\alpha\|+\|x\|)+|<y_k|x_\alpha-x>|$. Since $\{x_\alpha\}$ is order-bounded, there exist an $\alpha_0\in I$ and a real number $M>0$ such that $\|x_\alpha\|<M$ for every $\alpha\geqslant\alpha_0$. So, $\varepsilon>0$, there exists $k_0\in N$ such that $\|y_{k0}-l\|<\varepsilon/2(M+\|x\|)$. Moreover, as $\{x_\alpha\}$ is $\sigma(S, D(S))$

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- convergent to x, there exists $\bar{\alpha} \in I$ such that $|<y_{k0} \mid x_\alpha - x>| < \varepsilon/2$ for every $\alpha \geqslant \bar{\alpha}$.

Let $\alpha_1 \geqslant \{\alpha_0, \bar{\alpha}\}$. Then for every $\alpha \geqslant \alpha_1$ we have

$$|<l, x_\alpha - x>| \leqslant \|l - y_{k0}\|(\|x_\alpha\| + \|x\|) + |<y_{k0}|x_\alpha - x>| < \frac{\varepsilon}{2(M + \|x\|)}(H + \|x\|)$$

$$+ \frac{\varepsilon}{2} = \varepsilon, \text{ i.e. } \lim <l, x_\alpha> = <l, x> .$$

REMARK. The statment (b) has a converse when H is finite-dimensional (i. e. in a finite-dimensional Hilbert space every weakly convergent net is order-bounded), but the author ignores whether this property characterizes any Hilbert space .

3. EXAMPLES

EXAMPLE 1. We denote by H a separable, infinite-dimensional complex Hilbert space, by $\{u_n\}$ an orthonormal basis fixed in H and by $S = Sp(\{u_n\})$ the linear manifold spanned by $\{u_n\}$.

Let $E_k : H \to H$ be the projection on the finite-dimensional subspace $S^k := Sp(u_1, \ldots, u_k)$:

$$E_k x := \sum_{j=1}^{k} <u_j \mid x> u_j \quad \forall x \in H .$$

It is straightforward to verify that the functional $<1>_k : S \times S \to C$ defined by

$$<x \mid y>_k := <E_k x \mid E_k y> = \sum_{j=1}^{k} <x \mid u_j><u_j \mid y>$$

is a degenerate inner-product on S. We shall denote S_k the space S endowed with the degenerate inner product $<1>_k$.

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The function $p_k : x \to p_k(x) := (\sum_{j=1}^{k} |<u_j \mid x>|^2)^{1/2}$ is a seminorm on S.

Of course for the family of seminorms $F(\{u_n\}) = \{p_k : k \in N\}$ holds $p_k(x) \leqslant p_{k+1}(x)$ $\forall k \in N$ and $\forall x \in S$. In the previous Section we have endowed the pre-Hilbert space S with the weak topology $\sigma(S, D(S))$, that is of the topology generated by the family of seminorms $F(D(S)) = \{q_k := |<x \mid .>|, k \in S\}$.

We show now that in the present example we have that

$F(D(S))$ is equivalent to the totally ordered family of seminorms $F(\{u_n\})$.

PROOF. For every q_x, $x \in S$, we can write $x = \sum_{j=1}^{n} \alpha_j u_j$ and so for every $y \in S$,

$$q_x(y) = |<\sum_{j=1}^{n} \alpha_j u_j \mid y>| \leqslant \sum_{j=1}^{n} |\alpha_j| \, |<u_j \mid y>| .$$

If $A = max(|\alpha_j|)_j$ we have that $q_x(y) \leqslant A \sum_{j=1}^{n} |<u_j \mid y>| \leqslant A \sum_{j=1}^{n} p_j(y)$.

Moreover, $(p_k(x))^2 = \sum_{j=1}^{k} |<u_j \mid x>|^2 \leqslant (\sum_{j=1}^{k} |<u_i \mid x>|)^2$

$= (\sum_{j=1}^{k} q_{u_j}(x))^2 \Rightarrow p_x(x) \leqslant \sum_{j=1}^{n} q_{u_j}(x)$.

Finally, we construct S'. We recall that

$$S' = \{ t : S \to C : \exists \{x_n\} \in L_w(H^*, S) \text{ such that } <t, y> = \lim_n <x_n \mid y> \ \forall y \in S \} .$$

Since $S = Sp(\{u_n\})$, we have that

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Let $t \in S'$. If $a_j = \langle t | u_j \rangle \ \forall j \in N$ and $y \in S$, $y = \sum_{i=1}^{r} \lambda_i u_i$, then

$$\langle t, y \rangle = \sum_{i=1}^{r} \lambda_i a_i .$$

The (*)[1] suggest to put, *formally*, $t = \sum_{j=1}^{\infty} a_j u_j$, so we can identify t with the sequence $\{a_j\}$.

The map $\phi : S' \to C^N$, *defined for every* $t = \sum_{j=1}^{\infty} a_j u_j$ *in* S' *by* $\phi(t) = \{a_j\}$ *is a linear isomorphism between* S' *and* C^N .

In fact it is easy to verify that ϕ is linear and injective . If $\{\alpha_j\} \in C^N$ we consider the sequence $\{x_n\} \subseteq S$ defined by $x_n = \sum_{j=1}^{n} \alpha_j u_j$. It is easy to see that $\{x_n\} \in L_w (H^*, S)$ and if $t = w\text{-}\lim_n x_n$, *then* $\phi(t) = \{a_j\}$, *thus* ϕ is surjective .

EXAMPLE 2. If H is the Hilbert space $L_2 (R^n)$ of square Lebesgue-integrable complex-valued functions on R^n and S is the space $T(R^n)$ dense in $L_2 (R^n)$ of the functions of rapid decrease on R^n , then S' is the space of temperated distributions $T'(R^n)$ according to the definition in [5] . We have so obtained the iper-dual system $[T'(R^n), T(R^n)>$ and the Hilbert triplet $T(R^n) \subseteq L_2 (R^n) \subseteq T'(R^n)$.

EXAMPLE3. *Let* $H = l_2$ and $S = T$, where

$$T = \{ \{a_n\} \in l_2 : \Sigma \mid a_n \mid^2 (n+1)^m < \infty \ \forall m \in N \} .$$

We show that S' coincides with the pual of T endowed with a suitable topology . If. $m \in N$ is fixed, we define the norm $\| \ \|_m$ with

$$\| \underline{a} \|_m : = (\Sigma \mid a_n \mid^2 (n+1)^{1/2} \ \forall \ \underline{a} = \{a_n\} \in T .$$

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The family of norms $(\| \|_m)_{m \in N}$ is a directed family of norms, that is $\forall k_1, \ldots, k_r \in N$, $\exists\, k \in N$ and $\exists\, c \in R_+$ such that $\| \underline{a} \|_{k_1} + \ldots + \| \underline{a} \|_{k_r} \leqslant c \| \underline{a} \|_k \ \forall a \in T$ since, put $k = max\ (k_1, \ldots, k_r)$, we have $\| \underline{a} \|_{k_1} + \ldots + \| \underline{a} \|_{k_r} \leqslant r \| \underline{a} \|_k$.

We consider now the countably normed space $(T, \| \|_m)$. It is well known that a sequence $(\underline{a}^{(n)})_{n \in N}$ in $(T, \| \|_m)$ converges to zero iff $\lim_n \| \underline{a}^{(n)} \|_m = 0 \ \forall m \in N$. It is also easy to prove that $(T, \| \|_m)$ is a complete countably normed space.

The proof of following lemmas 1 and 2 are in [6] :

LEMMA 1. A linear functional $t : (T, \| \|_m) \to C$ is continuous iff $\exists c \in R_+$ and $\exists k \in N$ such that

$$| t (\underline{a}) | \leqslant c \| \underline{a} \|_k \ \forall \underline{a} \in T.$$

LEMMA 2. (UNIFORM BOUNDEDNESS PRINCIPLE IN COUNTABLY NORMED SPACES). Let X be a complete countably normed space with $\| \|_r$ a directed sequence of norms Let F be a set in X^*, the dual of X. If $\{ F((f) : F \in \mathbf{F} \}$ is bounded for each f X then there is a $c \in R_+$ and an $r \in N$ so that, for all $f \in X$ and all F $\mathbf{F}$,

$$| F(f) | \leqslant c \| t \|_r .$$

Utilizing the Lemmas 1 and 2 it is now easy to prove the

LEMMA 3. Let $(\underline{x}^{(n)} \subseteq l_2$ be a sequence in l_2 such that $\exists \lim_n < \underline{x}^{(n)} | \underline{a} > \ \forall \underline{a} \in T$. Then, if $t = w\text{-}\lim_n < \underline{x}^{(n)} |$, we have that $t \in (T, \| \|_m)'$, where $(T, \| \|_m)'$ is the topological dual of the countably normed space $(T, \| \|_m)$.

LEMMA 4. Let $(\underline{e}^{(n)})$ be the canonical basis in l_2 $(\underline{e}^{(n)} = \{ \delta_{kn} \}_{k \in N})$.

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Let $T \in (\mathbf{T}, \| \|_m)'$, $b_n = T(\underline{e}^{(n)})$. Then there is is a $c \in \mathbb{R}$ and a $k \in N$ so that $| b_n | \leqslant c (1+n)^m$ and

$$(*)^3 \quad T\underline{a} = \sum_{n=1}^{\infty} a_n b_n \quad \forall \underline{a} \in \mathbf{T},$$

Conversely, if $\| b_n \| \leqslant c (1+n)^m$, then $(*)^3$ defines an element of $(\mathbf{T}, \| \|_m)'$.

PROOF. If $T \in (\mathbf{T}, \| \|^3)'$, then from lemma 1 follows $| b_n | \leqslant c (1+n)^m$ and moreover the $(*)^3$ is trivial, Conversely, if $1\ b_n | \leqslant c (1+n)^m$ then

$$| \Sigma a_n b_n |^2 \leqslant (\Sigma | a_n |^2 (n+1)^{2m+2}) (\Sigma | b_n |^2 (n+1)^{-(2m+2)})$$
$$\leqslant 1/6\, c^2 \pi^2 \| \underline{a} \|^2_{2m+2}$$

and so, still from lemma 1, the $(*)^3$ defines an element of $(\mathbf{T}, \| \|_m)'$

THEOREM. In the previous notations, we have $S' = (\mathbf{T}, \| \|_m)'$,

PROOF. $S' \subseteq (\mathbf{T}, \| \|_m)'$ is the Lemma 3.

Conversely, if $T \in (\mathbf{T}, \| \|_m)'$, then from Lemma 4, if $b_n = T(\underline{e}^{(n)})$, it results $| b_n | \leqslant c (1+n)^m$ and $T(\underline{a}) = \Sigma a_n b_n$. But then, put $\underline{x}^{(n)} = (b_1, \ldots, b_n, 0, 0, \ldots)$ we have that $\forall \underline{a} \in T$, $\exists \lim_n \langle \underline{x}^{(n)} | \underline{a} \rangle = \Sigma b_n a_n = T(\underline{a})$

We have thus obtained the iper-dual system $[(\mathbf{T}, \| \|_m)', \mathbf{T}>$ and the Hilbert triplet $\{ \{a_n\} : \Sigma_n a_n |^2 (n+1)^n < \infty \ \forall_m \}$
$\subseteq \{\{a_n\} : \Sigma | a_n |^2 < \infty\} \subseteq \{\{a_n\} : \exists c \in R_+, \exists m \in N, | a_n | \leqslant c(1+n)^m\}$.

REMARK. If $(\mathbf{T}(R), \| \|_{b,h})$ denotes the countably normed space of functions of rapid decrease on R with $\| \|_{k,h}$ defined by $\|\Phi\|_{k,h} := \| x^k \Phi^{(h)}(x) \|_\infty$, where $\| \Phi \|_\infty := \sup_{x \in R} | \Phi(x) |$, it is well known that

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the countably normed space ($T(R)$, $\| \|_{k,h}$) and ($T, \| \|_m$) are isomorphic ([6]) . This provide, "en passant", a proof of equivalence of definitions of tempered distributions via linear functionals ([f]) and via weakly Cauchy sequences [5] .

EXAMPLE 4. Let H be a separable Hilbert space , Let B (H) be the space of bounded operators on H . We are interested to the following linear varieties of B (H) :

$$F(H) := \{ \text{ finite rank operators } \}$$
$$HS(H) := \{ \text{ Hilbert-Schmidt operators } \}$$
$$K(H) := \{ \text{ compact operators } \}$$

We recall that [8] :

(1) $F \in F(H)$ iff the range $R(F)$ of F is finite-dimensional .

(2) $K \in K(H)$ iff $\exists \{ v_i \}$, $\{w_i\}$ orthonormal bases of H and $\exists \{\mu_i\} \subseteq R_+$ with $\mu_1 \geqslant \mu_i \ \forall_i \in N$ and $\lim_i \mu_i = 0$, so that

$$(*)_1^4 \qquad Kx = \sum_{i=1}^{\infty} \mu_i < x \mid v_i > w_i \quad \forall x \in H .$$

(3) $T \in HS(H)$ iff $T \in K(H)$ and the sequence $\{\mu_i\}$ of $(*)_1^4$ is in l_2 ,

(4) $T(H) \subset HS(H) \subseteq K(H) \subseteq B(H)$.

(5) If $T_1, T_2 \in HS(H)$ and $\{u_i\}$ is an orthonormal basis of H, the numerical series $\sum_{i=1}^{\infty} < T_1 u_i \mid T_2 u_i >$ is absolutely convergent and the sum is independent from the orthonormal basis $\{u_i\}$.

(6) $HS(H)$ can be endowed with the inner product $< T_1 \mid T_2 > := \Sigma < T_1 u_i \mid T_2 u_i >$ where $\{u_i\}$ is any orthonormal basis in H . With respect to this inner product, $HS(H)$ is a separable Hilbert space, which we denote by H.

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(7) If we denote with S the space F(H), it results that S is dense in H. So we can construct S' as in Section 2 .

On the contrary of example 1 in which beginning from the space S of finite sequences we have obtained that S is the space of *all* sequences, now, beginning from the space S of finite rank operators, we obtain that $S' \subset$ B (H) and so it is not made up of *all* linear operators on H, as it results from the following theorem :

THEOREM In the previous notations, we have $\mathbf{K(H)} \subseteq S' \subseteq \mathbf{B(H)}$.

PROOF. Let $K \in$ K(H) . Then $Kx = \sum_{1}^{\infty} \mu_l <x|v_l> w_l$, $\forall x \in$ H . We can look to K as a functional on S defined by

$$K(F) : = \quad = \sum_{l=1}^{s} \lambda_l <K \tilde{v}_l | w_l> \forall F \in S, Fx = \sum_{l=1}^{s} \lambda_l <x | \tilde{v}_l> w_l .$$

For each $n \in N$ we define $K_n \in H$ with

$$K_n X = \sum_{l=1}^{n} \mu_l <x | v_l> w_l ,$$

and we show that $K = \text{w } lim <K_n |$.

Indeed if $F \in S$, $Fx = \sum_{l=1}^{s} \lambda_l <x | \tilde{v}_l> w_l$,

$$(*)_2^4 \qquad <K_n | F> = \sum_{l=1}^{\infty} <K_n \tilde{v}_l | F \tilde{v}_l> = \sum_{l=1}^{s} \lambda_l < K_n \tilde{v}_l | w_l>$$

and so $\lim_n <K_n | F> = K(F)$. In this way we have proved that $\mathbf{K(H)} \subseteq S'$.

We show now that $S' \subseteq$ B(H) . Let $t = w\text{-}\lim_n <T_n \in HS$ (H) . From $(*)_2^4$

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it is easy to see that

$$(*)_3^4 \quad \exists \lim_n < T_n x \mid y > \ \forall x, y \in \mathbf{H}$$

(in effect this is a necessary and sufficient condition in order that $\exists \lim_n < T_n \mid F > \ \forall F \in S$) .

Of course the $(*)_3^4$ is equivalent to

$$(*)_4^4 \quad \exists \lim_n < T_n^* x \mid y > \quad \forall x, y \in \mathbf{H}$$

(we recall that $T \in HS(\mathbf{H}) \Rightarrow T^* \in HS(\mathbf{H})$. where T^* denotes the adjoint of T) .

Now from $(*)_3^4$, once fixed x, from the Uniform Boundedness Principle and from the Fréchet-Riesz theorem, it follows that $\exists ! Tx \in \mathbf{H}$ such that

$$\lim_n < T_n x \mid y > = < T x \mid y > \quad \forall y \in \mathbf{H} .$$

As x run over $\mathbf{H}$, we obtain a map $T : \mathbf{H} \to \mathbf{H}$ which is linear, as it is easy to verify .

Analogously from $(*)_4^4$ we obtain a linear mapping T satisfying

$$(*)_5^4 \quad \lim_n < T_n^* x \mid y > = < \tilde{T} x \mid y > \quad \forall x, y \in \mathbf{H}$$

There remain to prove that $T \in B(\mathbf{H})$. We recall that is sufficient to prove that T has an adjoint operator defined *on* $\mathbf{H}$, [4] .

This is immediately verified once noted that the adjoint of T is the operator $\tilde{T}$ since $< Tx \mid y > = \lim_n < T_n x \mid y > = \lim_n < x \mid T_n^* y >$

$= < x \mid \tilde{T} y > \quad \forall x, y \in \mathbf{H}$.

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Jñānābha, Vol. 18, 1988

ON SOME FIXED POINT THEOREMS

By

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(*Received : January 20, 1988*)

There has been given a series of extensions of the well-known Banach Contraction Principle. Caristi [1] gave an important theorem where the function is not of contraction type, not even continuous. In this paper we prove a theorem where the function is not necessarily continuous and then we derive several known results as corollaries .

We need the following terminologies .

Let X be a metric space and $f : X \to X$ be a self mapping . Then, for $x \in X$, $O(x,\infty)=(x, fx, f^2x \ldots\ldots)$ is called the orbit of x. X is said to be orbitally complete if each Cauchy sequence in $O(x,\infty)$ converges there. If X is a complete metric space then it is orbitally complete, however, an orbitally complete metric space is not necessarily complete .

Let us recall the following well-known results.

Theorem 1. Let $f : X \to X$ be a contraction map on a complete metric space. Then f has a nnique fixed point .

Theorem 2. Let $f : X \to X$ and $\Phi : X \to [0, \infty)$, where X is a complete metric space and Φ is a lower semicontinuous function. If for each $x \in X$

$$d(x, fx) \leqslant \Phi x - \Phi(fx)$$

then f has a fixed point.

Note that f is not even continuous in this case.

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We state our main result as follows .

Theorem 3. Let $f : X \to X$ be a function, where X is an orbitally complete metric space , Assume further that

(i) for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that if $d(x, fx) < \delta(\varepsilon)$ then $f [B(x, \varepsilon)] \subset B(x, \varepsilon)$, and

(ii) $d(f^n x_0, f^{n+1} x_0) \to 0$ as $n \to \infty$ for some $x_0 \in X$.
Then the sequence $\{ f^n x_0\}$ converges to a fixed point of f ,

Proof . Let $0(x, \infty) = \{ x, fx, f^2x, \quad . \}$. We want to show that $\{ f^n x_0\}$ is a Cauchy sequence .

Given $\varepsilon > 0$ we choose N such that $d(f^n x_0, f^{n+1} x_0) < \delta(\varepsilon)$ for all $n \geqslant N$. For simplicity, we set $x_n = f^n x_0$, for all $n \in N$.

Now $d(x_N, x_{N+1}) = d (x_N, fx_N) < \delta (\varepsilon)$ so we get $f [B(x_N, \varepsilon)] \subset B(x_N, \varepsilon)$. In particular $x_{N+1} \in B(x_N, \varepsilon)$. By induction, we get that $f_i x_N = x_{N+i} \in B(x_N, \varepsilon)$ for all $i \geqslant 0$. Thus $d(x_i, x_r) \leqslant 2\varepsilon$ for $i, r \geqslant N$. This shows that $\{ f^n x_0 \}$ is a Cauchy sequeuce . Since X is an orbitally complete metric space $\{f^n x_0\}$ converges to some x in X.

We show that $fx = x$.

If $x \neq fx$, then $d(x, fx) = \alpha > 0$. We seek a contradiction .

We choosen sufficiently large such that $x_n = f^n x_0 \in B(x, \alpha/3)$ and $d (x_n, x_{n+1}) < \delta(\alpha/3)$. So $f [B(x_n, \alpha/3)] \subset B(x_n, \alpha/3)$. This gives that $fx \in B(x_n, \alpha/3)$.

However,

$$d(x_n, fx) \geqslant d(fx, x) - d(x, x_n) \geqslant (2/3) \alpha .$$

So $fx \notin B(x_n, \alpha/3)$, a contradiction .

This implies that $d(x, fx) = 0$, *i. e.*, x is a fixed point of f ,

Now, we derive several known results as corollaries .

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Remark. If $f : X \to X$ is a contraction map, *i. e.*, $d(fx, fy) \leqslant kd(x, y)$ for any $x, y \in X$, where $0 \leqslant k < 1$, then (i) and (ii) of Theorem 3 are satisfied. Indeed, for any given $\varepsilon > 0$, let $\delta = (1-k)\varepsilon$. Now, if $d(x, fx) < \delta$, then for $z \in B(x, \varepsilon)$ we have that

$$d(fz, x) \leqslant d(fz, fx) + d(fx, x) \leqslant kd(z, x) + d(fx, x)$$
$$< k\varepsilon + (1-k)\varepsilon = \varepsilon$$

Hence (i) is satisfied.

Moreover,

$$d(f^n x, f^{n+1} x) \leqslant k^n d(x, fx) \qquad \text{for any } x \in X.$$

Hence (ii) is satisfied, too.

We close this note with the following result.

Corollary (see [2]). Let $f : X \to X$ satisfy the condition

$$d(fx, fy) \leqslant \Phi(d(x, y))$$

where X is an orbitally complete metric space, $\Phi : R^+ \to R^+$ is a monotone, increasing function such that $\Phi^n(t) \to 0$ for each fixed $t > 0$. Then f has a unique fixed point x, and $x_n = f^n x$ converges to x.

Proof. Clearly $d(f^n x, f^{n+1} x) \to 0$ as $n \to \infty$. Now, since $d(f^n x, f^{n+1} x) \leqslant \Phi^n(d(x, fx))$, let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon - \Phi(\varepsilon)$. If $d(x, fx) < \delta$ then for $z \in B(x, \varepsilon)$ we get $d(x, fz) \leqslant d(x, fx) + d(fx, fz) < \varepsilon - \Phi(\varepsilon) + \Phi(d(x, z))$. Being Φ monotone increasing, we have that $f(z) \in B(x, \varepsilon)$, so $f[B(x, \varepsilon)] \subset B(x, \varepsilon)$. Then the result follows from Theorem 3.

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Jñānābha, Vol. 18, 1988

FIXED AND COMMON FIXED POINTS IN CONVEX METRIC SPACES

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Recently many authors have proved fixed point theorems) see for example [1], [3], [5], [9]) for operators mapping a Banach space X into itself. In each of these theorems the operators involved are nonexpansive *i. e.* if T maps the Banach space X into itself, then

(a) $\| Tx-Ty \| \leqslant \| x-y \|$, for $x, y \in X$

Kannan [7], [8] proved some fixed point theorems for operators mapping a Banach space into itself which, instead of beiag nonexpansive possess the following : if T is a mapping of a Banach space X into itself, then

(b) $\| Tx-Ty \| \leqslant 1/2 \, [\| x-Tx \| + \| y-Ty \|]$ for $x, y \in X$.

It may be observed that condition (a) implies the continuity of the operaters in the whole space while condition (b) has no such implication.

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The main aim of the present paper is to generalize the results of Kannan [7], [8] to convex metric spaces. We also extend the results of Kirk [9], Goebel, Kirk and Shimi [6] to the setting of convex metric spaces.

In the sequel we use I for the iaterval $[0, 1]$. The following notion was introduced by Takahashi [11].

Definition 1.1. Let X be a metric space. A mapping $W : X \times X \times I \to X$ is said to be a *convex structure* on X if for all $x, y \in X$ and $\lambda \in I$ the following condition is satisfied :

$$d(u, W(x, y ; \lambda)) \leqslant \lambda d(u, x) + (1-\lambda) d(u, y)$$

for all u in X. We will call a metric space with a convex structure a *convex metric* space.

If X is a Banach space, then, as a metric space with $d(x, y) = \| x-y \|$, the mapping $W : X \times X \times I \to X$ defined by $W(x, y ; \lambda) = \lambda x + (1-\lambda) y$ is a convex structure,

Definition 1. 2. Let X be a convex metric space. Let K be a nonempty subset of X. we say that K is *convex* if $W(x, y ; \lambda)$ belongs to K for all x, y in K and $\lambda \in I$.

Definition 1. 3. Let X be a metric space and E be a nonempty subset of X, then we define

$$R_x(E) = sup\{ d(x,y) : y \in E \}$$

$$R(E) = inf \{ R_x(E) : x \in E \}$$

$$E_C = \{ x \in E : R_x(E) = R(E) \}.$$

Definition 1. 4. Let X be a convex metric space. We say that X has *property* (C) if every bounded decreasing net of nonempty closed and convex subsets of X has a nonempty intersection.

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It follows from Smulian's theorem that every weakly compact convex subset of a Banach space has property (C) [4, p. 433] .

Lemma 2.1. Let X be a convex metric space . If X has property (C) , then E_C is nonempty, closed and convex .

Proof . For each $n \in N$, let $C(x,n) = \{ y \in E : d(x, y) \leqslant R(E) + \frac{1}{n} \}$ and $F_n = \bigcap_{x \in E} C(x, n)$. It follows that $\{ F_n \}$ is a decreasing sequence of nonempty, closed and convex sets. Since X has property (C), $E_C = \bigcap_{n=1}^{\infty} F_n$ is nonempty, closed and convex .

Definition 2. 2. Let X be a metric space. A mapping $T : X \to X$ is said to satisfy *condition* (A) if for all $x, y \in X$ we have

$$d(Tx, Ty) \leqslant max\{ d(x, Tx) , d(y, Ty) , [d(x, Ty) + d(y, Tx)] /3, [d(x, y) + d(x, Tx) + d(y, Ty)] / 3 \} ,$$

The proof of the next theorem is similar to that used by Kirk [9] in Banach space . The theorem extends Kirk's theorem and Theorem 1 of Kannan [7] .

Theorem 2. 3. Let X be a convex metric space having property (C). Let D be a nonempty, bounded, closed and convex subset of X. Let $T : D \to D$ be a mapping staisfying condition (A) with the property

$$\sup_{z \in K} d(z, Tz) \leqslant \frac{\delta(K)}{2} ,$$

where K is any nonempty convex subset of D which is mapped into itself by T. Then T has a unique fixed point in D .

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Proof. Let H be the family of all nonempty, closed and convex subsets of D which is mapped into itself by T. By property (C) and Zorn's Lemma it follows that H has a minimal element, say E.

By Lemma 2.1, E_C is nonempty, closed and convex For $x \in E_C$ and $y \in E$ we have

$$d(Tx, Ty) \leqslant max \{ d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/3,$$

$$[d(x,y) + d(x, Tx) + d(y, Ty)] / 3 \}$$

$$\leqslant max \{ \frac{\delta(E)}{2}, [d(x, y) + d(y, Ty) + d(y, x) + d(x, Tx)] / 3,$$

$$[d(x, y) + \delta(E)] / 3$$

$$\leqslant max \{ \frac{\delta(E)}{2}, [2d(x, y) + \delta(E)] /3, [d(x, y) + \delta(E)] /3 \}$$

$$\leqslant sup \{ d(x, y) : y \in E \} = R_x(E) = R(E).$$

So, $T(E)$ is contained in a closed spherical ball $S(Tx, R(E))$ and hence $T(E \cap \bar{S}) \subseteq E \subseteq \bar{S}$ (as $T(E) \subseteq E$) By the minimality of E, we conclude that $E \subseteq \bar{S}$.

Hence,

$$d(Tx, y) \leqslant R(E)$$

for all y in E. So,

(1) $sup \{ d(Tx, y) : y \in E \} \leqslant R(E)$.

Hence,

$R_{T(x)}(E) = sup \{ d(Tx, y) : y \in E \} \leqslant R(E)$,
which implies that

(2) $R_{T(x)}(E) = R(E)$, *i. e.* $T(x)$ belongs to E_C. Therefore, we have proved that $T(E_C) \subseteq E_C$.

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We assert that, *if* E contains more than one element, then E_C is a proper subset of E. Suppose not, *i. e.* $E = E_C$. Then for x, y in E,

$$R_x(E) = R_y(E) = R(E).$$

Therefore, $sup\ \{ d(x, z) : z \in E \} = sup\ \{ d(z, y) : z \in E \}$

for all x, y iu E. From this it follows that $sup\ \{ d(x,z) : z \in E \} = R(E)$, a constant for all $x \in E$, Hence $\delta(E) = sup\{ d(x,z) : x, z \in E \} = R(E)$.

This implies that for $x \in E = E_C$ (and x in E_C)

(3) $sup\ \{ d(Tx, z) : z \in E \} = \delta(E)$.

For x, y in $E = E_C$, by the definition of T we have

$$d(Tx, Ty) \leqslant max\ \{ d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)] / 3, [d(x,y) + d(x, Tx) + d(y, Ty)] / 3 \}$$

$$\leqslant max\ \{ \frac{\delta(E)}{2}, 2/3\ \delta(E), [\delta(E) + \delta(E)] / 3 \}$$

$$\leqslant 2/3\ \delta(E).$$

Therefore $T(E_C)$ is contained in a closed spherical ball $S(Tx, 2/3\ \delta(E))$ and hence $T(E_C \cap \bar{S}) \subseteq E_C \cap \bar{S}$ (*as* $T(E_C) \subseteq (E_C)$. Then by the minimality of E, $E = E_C$, we comclude that $E_C \subseteq \bar{S}$. Hence $d(Tx, y) \leqslant 2/3\ \delta(E)$ for all y in E. So, $sup\ \{ d(Tx, y) : y \in E \} \leqslant 2/3\ \delta(E)$, which contradicts (3), because E contains more than one element. Thus, we conclude that if E contains more than one element, then E_C is a proper subset of E. But then in view of (2) it contradicts the minimality of E. Therefore, E contains only one element say v, Since T maps E into itself, v is a fixed point of T. The uniqueness of the fixed point v is obvious.

Theorem 1 [7] follows as a corollary of our theorem 2.3. Kirk [9] has proved a fixed poind theorem using theorem of Smulian and the concept of normal structure where, however, the unicity is not guaranteed. The following example shows that our result is indeed an extension of Kirk's theorem,

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Example 2.4. Let $X = [0, 7]$ with the usual metric. Define $T : X \to X$ as follows

$$Tx = \begin{cases} x/3 & 0 \leqslant x \leqslant 6 \\ -2x+14 & 6 \leqslant x \leqslant 7 \end{cases}$$

Then T satisfies condition (A) with 0 only fixed point, but T is not nonexpansive. Indeed, let $x=6$, $y=7$ then $d(x, y) = 1 < 2 = d(Tx, Ty)$.

Definition 2.5. Let X be a convex metric space. We will say that X is *strictly convex* if for any x, y in X and $\lambda \in I$, there exists a unique $z \in X$ such that $\lambda d(x, y) = d(z, y)$ and $(1-\lambda)\, d(x, y) = d(z, x)$.

Lemma 2.6. Let X be a strictly convex metric space and C be a nonempty convex subset of X. Let $T: C \to C$ be a mapping as in Theorem 3.2, then $F(T)$ is convex.

Proof. Let x, y be in $F(T)$; we need to show that for any λ, $0 \leqslant \lambda \leqslant 1$; $W(x, y; \lambda)$ belongs to $F(T)$. Since C is convex, $W(x, y; \lambda)$ belongs to C for x, y in $F(T)$ and $\lambda \in I$. Let $z = W(x, y; \lambda)$. Now

(1) $d(x, y) = d(Tx, Ty) \leqslant d(x, Tz) + d(Tz, y)$.

Using the definition of T we have

(2) $d(x, Tz) \leqslant max \{ d(x, z), [d(x, Tx) + d(z, Tz)]/2$

$[d(x, Tz) + d(z, Tx)]/2 \}$

$= max \{ d(x, z), d(z, Tz)/2, [d(x, Tz) + d(z, x)]/2 \}$

Now $d(x, Tz) \leqslant 1/2\, d(z, Tz) \leqslant 1/2 [d(z, x) + d(x, Tz)]$ imp/les $d(x, Tz) \leqslant d(x, z)$. Thus from (2) we have

(3) $d(x, Tz) = d(x, T(W(x, y; \lambda))) \leqslant d(x, z) =$

$d(x, W(x, y; \lambda))$;

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Similarly we wave

$$(4)\quad d(y, T(W(x, y; \lambda))) \leqslant d(y, W(x, y; \lambda)),$$

Using (3) and (4) we can write (1) as

$$d(x, y) \leqslant d(x, T(W(x, y; \lambda))) + d(T(W(x, y; \lambda)), y)$$
$$\leqslant d(x, W(x, y; \lambda)) + d(W(x, y; \lambda), y)$$
$$\leqslant \lambda d(x, x) + (1-\lambda) d(x, y) + \lambda d(y, x) + (1-\lambda) d(y, y)$$
$$= d(x, y).$$

Thus, we conclude that all inequalities are equalities,

i. e.

$$d(x, T(W(x, y; \lambda))) + d(T(W(x, y; \lambda)), y)$$
$$= d(x, W(x, y; \lambda)) + d(W(x, y; \lambda), y).$$

Hence

$T(W(x, y, \lambda)) = W(x, y; \lambda)$. Therefore $F(T)$ is convex.

Definition 3. 1. Let X be a metric space $T: X \to x$ be a mapping. T is said to have *property* (K) on $E \subset X$ *if*, for any closed subset F of E which contains more than one element and which is mapped into itself by T, there exists an $x \in F$ such that

$$d(x, Tx) < \sup_{y \in E} d(y, Ty).$$

Tkeorem 3. 2. Let X be a compact metric space and $T: X \to X$ be a continuous mapping satisfying the following condition :

$$d(Tx, Ty) \leqslant max \{ d(x, y), [d(x, Tx) + d(y, Ty)]/2,$$
$$[d(x, Ty) + d(y, Tx)]/2 \}$$

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for all $x, y \in X$. Suppose T has property (K) on X; then T has a fixed point.

Proof. Let H be the family of all nonempty, closed subsets of X which is mapped into itself by T. By Zorn's Lemma it follows that H has a minimal element D, If D consists of a single element, then that element must be a fixed point of T. Suppose D contains more than one element. Since T has property (K) over X, there exists a $z \in D$ such that

$$d(z, Tz) = \gamma < \sup_{y \in D} d(y, Ty),$$

Let $C = \{ x \in D : d(x, Tx) \leqslant \gamma \}$, Then C is nonempty proper subset of D. We claim C is closed too. Indeed, if $\{ x_n \}$ is in C and $x_n \to y$, then $y \in D$ (since D is closed).

By the triangle inequality, $d(y, Ty) \leqslant (y, x_n) + d(x_n, Tx_n) + d(Tx_n, Ty)$. The continuity of T implies $d(y, Ty) \leqslant \gamma$, so y is in C. It remains to show that C is mapped into itself by T. Let $x \in C$, then

$$d(Tx, T^2x) \leqslant max\{ d(xTx), [d(x, Tx) + d(Tx, T^2x)]/2,$$

$$[d(x, T^2x) + d(Tx, Tx)]/2 \}$$

Hence $d(Tx, T^2x) \leqslant d(x, Tx) \leqslant \gamma$. So C is mapped into itself by T. Thus C is nonempty, closed proper subset of D which is mapped into itself. This contradicts the minimality of D. Hence D consists of a single point, which must be a fixed point of T.

The following examples show that our theorem is indeed a generalization of the theorem of Kannan [7].

Example 3. 3. Let $X = R$ with the usual metric. Define $T : X \to X$ by $Tx = x$ for all $x \in R$. Then T satisfies condition of our mapping, but not that of Kannan.

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Example 3. 4. Let $X = [0, \infty)$ with the usual metric and $Tx = \frac{x}{1+x}$. Then T satisfies condition of our mapping, but not that of Kannan. In fact, for $x \neq y$, $d(Tx, Ty) = \frac{(x-y)(xy-1)}{xy} < d(x, y)$, os T satisfies our condition Let $x = 0$, $y = 1$, then $d(Tx, Ty) = 1/2 > 1/4 = 1/2\,[d(x, Tx) + d(y, Ty)]$.

Theorem 3. 5. Let X be a strictly convex metric space with property (C). Let D be a nonempty, bounded, closed and convex subset of X. Let $F : D \to D$ be a family of continuous commuting mappings with nonempty fixed points set and satisfying condition of Theorem 3. 2. Then F has a common fixed point in D.

Proof : It follows from Lemma 2 6 that $F(T)$, the fixed points set of T in F, is convex. By the continuity of T we infer that $F(T)$ is closed.

Let F_α be the fixed point set of $T\alpha$ in F; F_α is nonempty, closed, bounded ard convex. If $u \in F_\alpha$, then for any β, $T_\alpha T_\beta = T_\beta T_\alpha u = T_\beta u$; i. e. $T_\beta u$ lies in F_α and each T_β maps F_α into itself. If we are given a finite sequence $\alpha_1, \alpha_2, \ldots, \alpha_m$, and consider T_{α_m} as a mapping satisfying condition of theorem of $F_{\alpha_1} \cap F_{\alpha_2} \cap \ldots \cap F_{\alpha_m}$ into itself, it follows from the hypothesis that $\bigcap_{k=1}^{m} F_{\alpha_k} \neq \phi$. Since X has property (C), the family $\{F_\alpha\}$ has a nonempty intersection, but this consists of precisely the common fixed points.

Definition. Let B be a bounded set in a metric space X, and let $\delta(B)$ be its diameter. A point $x \in B$ is said to be a diametral point of B if $\sup_{y \in B} d(x, y) = \delta(B)$. A convex subset K of X is said to have

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normal structure if every bounded convex subset K_1 of K which contains more than one point, has a point that is not a diametral point of K_1.

Any compact convex metric space has normal structure [11, Proposition 5]. Every compact convex set of Banach space has a normal strueture [2 ρ. 6] ; so does every closed bounded and convex subset of a uniformly convex Banach space [10, p. 8].

Corollary 3. 6. Suppose X is strictly convex with property (C). Let K be a nonempty, bounded, closed and convex subset of X with normal structure If F is commuting family of nonexpansive mappings, of K into itself, then the family has a common fixed point.

As our final result, we prove an extension of the results of Goebel, Kirk and Shimi [6].

Definition 4. 1. Let X be a metric space, $T : X \to X$ be a mapping. T is said to have *property* (S) on $K \subset X$ if for any nonempty, closed bounded convex subset D of K which contains more than one element and which is mapped into ltself by T, we have

$$\sup_{y \in D} d(y; Ty) < \delta(D),$$

where $\delta(D)$ is the diameter of D.

Definition 4. 2 Let X be a metric space and $T: X \to X$ be a mapping. We say that T satisfies *condition* (GKS) if for all $x, y \in X$ we have

$$d(Tx, Ty) \leqslant a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx),$$

where $a_1, a_4, a_5 \geqslant 0$, $a_2, a_3 > 0$ and $\sum_{1}^{5} a_i = 1$.

Theorem 4.3. Let X be a convex metric space having property (C).

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Let K be a nonempty closed, bounded convex subset of X. Let $T : K \to K$ be a mapping staisfying condition (GKS) and having property (S) on X. Then T has a unique fixed point.

Proof : Let F be the family of all nonempty closed bounded convex subsets of K which are mapped into itself. Since $K \in F$, $F \neq \Phi$, using property (C) and Zorn's Lemma we can find a minimal subset D of K in F. Without loss of generality we may assume that $a_2 = a_3$ and $a_4 = a_5$. Let

$$\gamma(D) = (a_1 + a_4 + a_5) \sup_{x,y \in D} d(x, y) + (a_2 + a_3) \sup_{y\ D} d(y, Ty).$$

Then $\gamma(D) < (a_1 + 2a_4)\,\delta(D) + 2a_2\,\delta(D) \leqslant \delta(D)$, since $a_2 > 0$. We claim that D contains one point. Suppose not, let $x, y \in D$, $x \neq y$.

Then,

$$\begin{aligned} d(Tx, Ty) &\leqslant a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) \\ &\quad + a_4 d(x, Ty) + a_5 d(y, Tx) \\ &\leqslant (a_1 + 2a_4) \sup_{x,y \in D} d(x, y) + 2a_2 \sup_{y \in D} d(y, Ty) \\ &= \gamma(D). \end{aligned}$$

Thus $T(D)$ is contained a closed spherical ball C with center Tx and radius $\gamma(D)$.

It is obvious that $T(D \cap C) \subseteq D \cap C$. By the minimality of D we obtain $D \subseteq C$. Therefore $d(Tx, y) \leqslant \gamma(D)$ for each $y \in D$ and hence $\sup_{y \in D} d(Tx, y) \leqslant \gamma(D)$ for any arbitrary but fixed $x \in D$. Let

$$D_1 = \{ z \in D : \sup_{y \in D} d(z, y) \leqslant \gamma(D) \}$$

Then D_1 is nonempty, bounded closed and convex. Moreover

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$T(D_1) \subset D_1$. Also,

$$\delta(D_1) = \sup_{x,y \in D_1} d(x,y) \leqslant \sup_{\substack{y \in D \\ x \in D_1}} d(x,y)$$

$$\leqslant \sup_{x \in D_1} \sup_{y \in D} d(x,y) \leqslant \gamma(D)$$

$$< \delta(D) .$$

Thus, D_1 is a proper subset of D . This contradicts the minimality of D. Consequently D has only one point which is a fixed point of T in K. The unicity of fixed point is clear .

Remark 4. 4. A similar theorem for uniformly convex Banach was proved by Goebel, Kirk, and Shimi [16] without the restriction that $a_2, a_3 > 0$, however there the continuity of the mapping T was assumed ; moreover the fixed point was not unique ,

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Jñānābha, Vol. 18, 1988

FIXED POINTS FOR MULTI - VALUED MAPPINGS IN LOCALLY CONVEX SPACE

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(*Received : March 8, 1988*)

ABSTRACT

Let Q be a family of continuous seminorms determining the topology of a locally convex topological vector space E. In this paper, sufficient conditions are given for the existence of fixed points for multivalued mappings . The results of this paper strengthen those of Kirk, Assad and Kirk Markin, and others . Our results are motivated by the results of Samanta (12) and extend his results to locally convex spaces .

1. INTRODUCTION

Let X be a metric space and (2^X H) be the space of all nonempty compact subsets of X endowed with the Hausdorff metric H. Let $T : X \to 2^X$ be a mvltivalued mapping . A point $x \epsilon X$ is said to be a fixed point of T, provided $x \in T(x)$. Kirk [3] proved the following :

* Research supported in part by a grant from NSERC (Canada) .

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If K is a nonempty, closed, bounded, and convex subset of a reflexive Banach space X and if K possesses normal structure, then every self-nonexpansive mapping of K has a fixed point, Markin [8] extended the result of Kirk to multivalued mappings. Markin's result is : Let X be a real Hilbert space and $T : X \to 2^X$ be nonexpansive. If

$T(x) \subset B$ for x $\in B$, where $B = \{ x \in X ; \| x \| \leqslant 1 \}$,

then T has a fixed point .

Assad and Kirk [1] have extended this result of Markin to Banach spaces satisfying Opial's condition. Fixed points for set-valued non-expansive mappings and for set-valued Kannan-type mappings in locally convex spaces were given by Su and Sehgal [11] and Ko and Tsai [5], respectively. Our results, which are primarily motivated by the work of Ko [6], Ko and Tsai [5], and Su and Sehgal [11], extend the results of Assad and Kirk [1], Kirk [4], Markin [8], Nadler [9], and others .

2. PRELIMINARIES AND NOTATIONS

Throguhout this paper, E will denote a Hausdorff, locally convex, linear topological space and Q, a (fixed) family of continuous semi-norms whic hgenerate the topology of E. For any $p \in Q$ and $A \subseteq E$, $\delta_p(A)$ will denote the p-diameter of A, that is, $\delta_p(A) = sup \{ p(x-y) : x, y \in A \}$, $B_p(x, r)$ ($B_p[x, r]$) will denote the open (closed) ball with center at X and radius r. For $A \subseteq E$, $d_p(x, A) = inf \{ p(x-y) : y \in A \}$. We shall denote by 2^E, the space of all nonempty compact subsets of E.

Definition 2. 1 A convex subset K of E is said to have *normal structure* if for every closed, bounded, convex subset W of K such that $\delta_p(W) > 0$, there exists $x \in W$ such that $sup \{ p(x-y) : y \in W \} < \delta_p(W)$.

Definition 2.2 Let K be a nonempty subset of E, and let $\{W_\alpha : \alpha \in \triangle\}$ be a

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decreasing set of nonempty bounded subsets of K. For $x \in K$, $\alpha \in \triangle$, and $p \in Q$, define

$$r_{p,\alpha}(x) = sup \{ p(x - y) : y \in W_\alpha \}$$
$$r_p(x) = inf \{ r_{p,\beta}(x) : \beta \in \triangle \}$$
$$r_p = inf \{ r_p(x) : x \in K \}$$
$$M = \{ x \in K : r_p(x) = r_p \} .$$

The set M and the number r_p will be called the asymptotic center and the asymptotic radii, respectively, of $\{ W_\alpha : \alpha \in \triangle \}$ in K with respect to p.

Definition 2.3 Let K be a nonempty subset of E. $T : K \to 2^K$ is said to be a *nonexpansive map* if for each $p \in Q$, for each $x, y \in K$, and for each $x_1 \in T(x)$, there exists $y_1 \in T(y)$ such that $p(x_1 - y_1) \leqslant p(x - y)$.

Note that if the set K is compact, then each set A in 2^K is compact and, hence, bounded under each $p \in Q$.

Thus, the Hausdorff metric D_p [2] induced from the seminorm p is well defined on 2^K. It should be noted that D_p is a metric on 2^K, although p is a seminorm on K. Hence, if K is compact, Definition 2.3 is equivalent to the following :

Definition 2.4 Let K be a compact subset of E. A map $T : K \to 2^K$ is said to be a nonexpansive map if for each $p \in Q$ and $x, y \in K$, we have $D_p(Tx, Ty) \leqslant p(x - y)$.

3. RESULTS

Theorem 3.1 Let E be a semireflexive, locally convex Hausdorff topological vector space and K be a closed, bounded, convex subset of E, possessing normal structure. If $T : K \to 2^E$ is a mapping such that

(i) $T(x) \cap K = \phi$ for all $x \in K$,

(ii) for any closed convex subset L of K satisfying

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$T(w) \cap L = \phi$ for all $w \in L$,

$D_p(T(x) \cap L, T(y) \cap L) \leqslant p(x-y)$ whenever

$x, y \in L$, $x \neq y$. Then T has a fixed point.

Proof Let G be the family of all nonempty, closed convex subsets M of K such that $T(x) \cap M \neq \phi$ for all $x \in M$. Then, G is nonempty, since $K \in G$. Partially order G by inclusion. Let $F = \{F_i\}_{i \in \Delta}$ be a decreasing chain in G. It follows that $\bigcap_{i \in \Delta} F_i \neq \phi$ (due to the semireflexivity of E). Let $F_0 = \bigcap_{i \in \Delta} F_i$, a nonempty, closed convex subset of M. Let $x \in F_0$. By hypothesis, $T(x) \cap (\bigcap_{i \in \Delta} F_i) \neq \phi$. For any $F_i \in F$, $T(x) \cap F_i$ is compact and $\{T(x) \cap F_j\}_{j \geqslant i}$ is a family of nonempty closed subsets of the compact set $T(x) \cap F_i$ having finite intersection property. Consequently, $\bigcap_{j \geqslant i} (T(x) \subset F_j) \neq \phi$ and, therefore, $\bigcap_{i \in \Delta} (T(x) \cap F_i) \neq \phi$, *i. e.*, $T(x) \cap F_0 \neq \phi$. Thus, any chain in G has a greatest lowerbound, and by Zorn's Lemma, there is a minimal member N in G.

We claim that N is a singleton set. If not, then as shown by Lim [7, p 315–316], the center of N, denoted by R is a nonempty proper closed convex subset of N. It is enough to show that R belongs to G. Let $y \in R$, then $y \in N$ and $T(y) \cap N \neq \phi$. Take $z \in T(y) \cap N$. Let $W = B_p[z, r(N)] \cap N$ Then, W is a nonempty closed convex subset of N. Let $x \in W$. Using Hypothesis (ii), we have $D_p(T(x) \cap N, T(y) \cap N) \leqslant p(x-y)$.

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Since $z \in T(y) \cap N$, there is a $u \in T(y) \cap N$ such that

$p(u-y) \leqslant D_p(T(x) \cap N, T(y) \cap N) \leqslant p(x-y)$. Hence,

$T(x) \cap W \neq \phi$ for all $x \in W$. By the minimality of N, we have

$N \subseteq W$, that is, $N \subseteq B_p[z, r(N)]$. Consequently, $z \in R$ and

we have $T(y) \cap R \neq \phi$ for all $y \in R$. Therefore $R \in G$, a contradiction to the minimality of N. Hence, N consists of a single point which is a fixed point of T.

The following example shows that Condition (ii) cannot be replaced by $D_p(Tx, Ty) \leqslant p(x-y)$ for all $x, y \in K$, $x \neq y$.

Example 2.1 Let $X = R^2$ with usual norm and

$K = \{(x,0) : 0 \leqslant x \leqslant 1\}$. Then, K is a bounded closed convex subset of a reflexive Banach space X. Define $T: K \to 2^X$ as follows :

$$T(x,0) = \{(\frac{-1}{\sqrt{2}}(\tfrac{1}{2} - x), \frac{1}{\sqrt{2}}(\tfrac{1}{2} - x)), (x + \tfrac{1}{2}, 0)\} \; 0 \leqslant x \leqslant \tfrac{1}{2},$$

$$T(x + \tfrac{1}{2}, 0) = \{(1 - \frac{x}{\sqrt{2}}, \frac{x}{\sqrt{2}}), (x, 0)\}, 0 \leqslant x \leqslant \tfrac{1}{2},$$

Then $T(x)$ intersects K for every x in K. If

$(x, 0), (y, 0) \in K$ such that $x, y \in [0, \frac{1}{2}]$ or $x, y \in [\frac{1}{2}, 1]$,

then $D(T(x,0), T(y,0)) \leqslant \|(x,0) - (y,0)\|$. If

$x, y \in [0, \frac{1}{2}]$, then $D(T(x,0), T(y + \frac{1}{2}, 0))$

$$\leqslant [(\frac{1}{\sqrt{2}}(\tfrac{1}{2} - x) - y)^2 + (\frac{1}{\sqrt{2}}(\tfrac{1}{2} - x)^2]^{1/2}$$

$$= [x^2 + y^2 + \sqrt{2}xy - (x + \sqrt{\frac{y}{2}} + \frac{1}{4}]^{1/2},$$

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whenever $(\sqrt{2}+2)xy \leqslant (1+\frac{1}{\sqrt{2}})y$, *i.e.*, $x \leqslant \frac{1}{2}$ with $y \neq 0$.

Thus, $D(T(x,0), T(y,0)) \leqslant \|(x,0)-(y,0)\|$ for all $(x,0), (y,0)$, $x \neq y$ in K. Now for $x, y \in [0, \frac{1}{2}]$,

$$D(T(x,0) \cap K, T(y+\tfrac{1}{2},0) \cap K) = \|(x+\tfrac{1}{2},0)-(y,0)\|$$
$$= \|x+\tfrac{1}{2}-y\|$$
$$= \|y+\tfrac{1}{2}-x\|$$

whenever $x > y$. So T is nonexpansive and satisfies all but Condition (ii) of Theorem 2.1. Clearly, T has no fixed point.

For the proof of Theorem 2.2, we need to recall the following :

Lemma 2.1 [10, Lemma 3.1]. Let H and K be two closed subsets of E such that $H \cap K \neq \phi$. If H is convex, then

$\partial_H(K) = \phi$ if and only if $H \subseteq K$ (here,

$\partial_H(K)$ denotes the relative boundary of $H \cap K$ in H).

Theorem 2.2 Let E be a Housdorff, locally convex topological vector space and K be a nonempty, closed convex subset of E. Let L be a nonempty, weakly compact, convex subset of K possessing normal structure and $T : L \to 2^E$
be a mapping satisfying $T(x) \cap K \neq \phi$ for all $x \in L$. Further, for M a nonempty, closed convex subset of K such that $T(x) \cap M \neq \phi$, whenever $X \in M \cap L$, the following conditions hold :

(i) $D_p(T(x) \cap M, T(y) \cap M) \leqslant p(x-y)$ for all
$x, y \in M \cap L$, for all $p \in Q$, and

(ii) $T(W) \cap (M \cap L) \neq \phi$, where $w \in \partial_M(L)$.

Then, T has a fixed point.

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Proof Let G be the family of all nonempty closed convex subsets M of K such that $M \cap L \neq \phi$ and $T(x) \cap M \neq \phi$ whenever $x \in M \cap L$. Then G is nonempty, since $K \in G$. Partially order G by inclusion. Arguing similarly, as in the proof of Theorem 2.1, it follows that every chain in G has a greatest lower bound. So, by Zorn's Lemma, there is a minimal element N in G. If $\partial_N(L) = \phi$, then, by Lemma 2.1, $N \subseteq L$. In this case, Theorem 2.1 is applicable, and T has a fixed point. Thus, we assume that $\partial_N(L) \neq \phi$ and show that $N \cap L$ is a singleton set. Let $\delta_P(N \cap L) > 0$, for some p.

Since L has normal structure, there is a point $y \in L \cap N$ such that

$$0 < \alpha = \sup_{x \in N \cap L} p(x - y) < \delta(N \cap L). \qquad (1)$$

Consider the set $H = \{ x \in N : N \cap L \subseteq B_P[x, \alpha] \}$, a closed convex subset of N. Since $y \in H$, H is nonempty and, by (1) $H \neq N$. We show that $H \in G$. Let $z \in H \cap L$ and $y \in \partial_N(L)$. Then, $y \in N \cap L$ and, hence, $p(z - y) \leqslant \alpha$. Now, since $N \in G$ from Hypothesis (i), we have, for $y, z \in N \cap L$,

$$D_p(T(y) \cap N, T(z) \cap N) \leqslant p(y - z). \qquad (2)$$

Consequently, $D_P(T(y) \cap N, T(z) \cap N) \leqslant \alpha$. Since $y \in \partial_N(L)$, by Hypothesis (ii), $T(y) \cap (N \cap L) \neq \phi$.

Take $u \in T(y) \cap (N \cap L)$. Then, using (2), we obtain for $v \in T(z) \cap N$, $p(u - v) \leqslant D_P(T(z) \cap N, T(y) \cap N) \leqslant p(z - y)$.

Let $W = B_P[v, \alpha] \cap N$. Then, W is a closeed convex subset of K with $W \cap L \neq \phi$ (since $u \in W \cap L$). If $w \in W \cap L \cap N \cap L$, then $p(z - w) \leqslant \alpha$.

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Further, $D_p(T(W) \cap N, T(z) \cap N) \leqslant p(z-w) \leqslant \alpha$.

Since $v \in T(z) \cap N$, there exists an element $s \in T(w) \cap N$ such that

$p(v-s) \leqslant D_p(T(w) \cap N, T(z) \cap N) \leqslant p(w-z) \leqslant \alpha$.

This shows that $s \in W$, or that $T(w) \cap W \neq \phi$ for all

$w \in W \cap L$. Therefore, $W \in G$. By the minimality of N, we have $N = W$. Hence, $N \cap L \subseteq B_P[v, \alpha]$ and $v \in H$.

We now have $T(z) \cap H \neq \phi$ for all $z \in H \cap L$, and it follows that $H \in G$. But H is a proper subset of N, a contradiction. Hence, $N \cap L$ is a singleton set. Let $N \cap L = \{x\}$.

By assumption, $\phi \neq \partial_N(L \subseteq N \cap L, \partial_N(L) = \{x\}$.

Now, by Hypothesis (ii), $x \in T(x)$. This completes the proof.

Remark 2.1 The example below shows that Condition (ii) of Theorem 2.2 cannot be replaced by the following coudition :

(ii)' If $x \in \partial_K(T)$, then $T(x) \cap T \neq \phi$.

Example 2.2 Let $X = R^2$ with the usual norm and

$K = \{(x, 0) : 0 \leqslant x < \infty\}$ and

$L = \{x, 0) : 0 \leqslant x \leqslant 1\}$. Define $T : L \to 2^x$

by $T(x,0) = \{(1 + x, 0), (0, 1 - x)\}, (x\ 0) \in L$.

Tben K is a closed and convex subset of a reflexive Banach space, and L is a closed, bounded convex subset of K. The mapping T satisfies the condition $T(x) \cap K \neq \phi$ for all $x \in L$, Condition (i) of Theorem 2.2 and Condition (ii)' are also satisfied. By taking $M = [1/2, \infty)$, we see that Hypothesis (ii) of Theorem 2.2 is not satisfied. Clearly, T does not have a fixed point.

Remark 2.2 Our Theorem 2.2 is applicable to $L_p[0,1]$, $1 < p < 2$. However, the results of Markin, and Assad and Kirk, do not hold there, since $L_p[0, 1]$ is not a Hilbert space, nor does it satisfy Opial's condition.

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Jñānābha, Vol. 18, 1988

ON SOME FINITE INTEGRALS INVOLVING THE MULTIVARIABLE H - FUNCTION

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(Received : May 31, 1985 ; Revised [Final] : July 6, 1987)

ABSTRACT

In this paper, we evaluate some finite integrals involving the products of Jacobi polynomials and the H-function of several complex variables .

1. INTRODUCTION

Srivastava and Panda [13] have defined the multivariable H-function as (see also [12] and [14])

$$H\begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix} = H \begin{matrix} O,\varepsilon : (M', N') ; \ldots ; (M^{(r)}, N^{(r)}) \\ A, C : (B', D') ; \ldots : (B^{(r)}, D^{(r)}) \end{matrix} \begin{pmatrix} [(a) : \theta', \ldots, \theta^{(r)}] : \\ [(c) : C', \ldots, C^{(r)}] : \\ [(b') : \phi'] ; \ldots ; [(b'^{r}) : \phi^{(r)}] ; z_1 \\ [(d') : \delta'] ; \ldots ; [(d^{(r)}) : \delta^{(r)}] ; z_r \end{pmatrix}$$

$$= \frac{1}{(2\pi i)^r} \int_{L_1} \cdots \int_{L_r} V(s_1, \ldots, s_r)\, U_1(s_1) \ldots U_r(s_r)\, z_1^{s_1} \ldots z_r^{s_r} \cdot ds_1 \ldots ds_r \quad \ldots (1.1)$$

where

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$$V(s_1, \ldots, s_r) = \frac{\prod_{j=1}^{\varepsilon} \Gamma\left(1 - a_j + \sum_{i=1}^{r} \theta_j^{(i)} s_i\right)}{\prod_{j=\varepsilon+1}^{A} \Gamma_{(a_j} - \sum_{i=1}^{r} \theta_j^{(i)} s_i) \prod_{j=1}^{C} \Gamma(1-c_j + \sum_{i=1}^{r} C_j^{(i)} s_i)} \quad \ldots (1.2)$$

and

$$U_i(s_i) = \frac{\prod_{j=1}^{M^{(i)}} \Gamma\left(d_j^{(i)} - \delta_j^{(i)} s_i\right) \prod_{j=1}^{N^{(i)}} \Gamma\left(1-b_j^{(i)} + \phi_j^{(i)} s_i\right)}{\prod_{j=M^{(i)}+1}^{D^{(i)}} \Gamma\left(1-d_j^{(i)} + \delta_j^{(i)} s_i\right) \pi_{j=N^{(i)}+1}^{B^{(i)}} \Gamma\left(b_j^{(i)} - \phi_j^{(i)} s_i\right)}$$

$$i = 1, \ldots, r \quad \ldots (1.3)$$

The multiple integral in (1.1) absolutely converges if

$$|\arg(z_i)| < \tfrac{1}{2} T_i \pi, \; i = 1, \ldots, r \quad \ldots (1.4)$$

where

$$T_i = -\sum_{j=\varepsilon+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{N^{(i)}} \phi_j^{(i)} - \sum_{j=N^{(i)}+1}^{B^{(i)}} \phi_j^{(i)} - \sum_{j=1}^{C} C_j^{(i)}$$

$$+ \sum_{j=1}^{M^{(i)}} \delta_j^{(i)} - \sum_{j=M^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} > 0, \; i = 1, \ldots, r \quad \ldots (1.5)$$

The following notations and known results will be used throughout this paper :

$$(\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} = \begin{cases} 1, \text{ if } n = 0, \\ \alpha(\alpha+1)\ldots(\alpha+n-1), \end{cases} \quad \forall n \in \{1,2,3, \ldots\}.$$

$$\ldots (1.6)$$

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$$\sum_{R=0}^{k} \frac{(-k)_R\ (1+\alpha+\beta+k)_R}{(1+\beta)_R}\ \frac{t^R}{R!}$$

$$H \begin{matrix} 0,\ \varepsilon+2 : (M', N'); \ldots; (M^{(r)}, N^{(r)}) \\ A+2,\ C+2 : (B', D'); \ldots, (B^{(r)}, D^{(r)}) \end{matrix}$$

$$\left(\begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}], (-\eta : h_1, \ldots, h_r), (\beta-\eta : h_1, \ldots, h_r) : \\ [(c) : C', \ldots, C^{(r)}], (-\alpha-\eta-R-1 : h_1, \ldots, h_r), (\beta-\eta+R : h_1, \ldots, h_r) : \end{matrix} \right.$$

$$\left. \begin{matrix} [(b') : \phi'] ; \ldots ; [(b^{(r)}) : \phi^{(r)}] ; z_1 2^{h_1} \\ [(d') : \delta'] ; \quad . ; [(d^{(r)}) : \delta^{(r)}] , z_r 2^{h_r} \end{matrix} \right) \qquad \ldots (2.1)$$

where

$$\mathrm{Re}\,(\eta) > -1,\ \mathrm{Re}\,(\alpha) > -1,$$

$h_i > 0,\ i = 1, \ldots, r$ and $|\arg(z_i)| < \frac{1}{2} T_i \pi,\ i = 1, \ldots, r,$

$T_i > 0$ and $\mathrm{Re}\left(\eta + \sum_{i=1}^{r} h_i\, d_j^{(i)} / \delta_j^{(i)}\right) > -1.$

$$\int_{-1}^{1} \rho^k (1-x)^{\eta} (1-x)^{\alpha} P_k^{(\alpha,\alpha)} \left(\frac{1-xt}{\rho}\right) H \begin{pmatrix} z_1 (1+x)^{h_1} \\ \vdots \\ z_r (1+x)^{h_r} \end{pmatrix} dx$$

$$= \frac{2^{\eta+\alpha+1}\,\Gamma(1+\alpha+k)}{k!} \sum_{R=0}^{k} \frac{(-k)_R}{R!}\, t^R$$

$$H \begin{matrix} 0,\ \varepsilon+2 : (M', N'); \ldots; (M^{(r)}, N^{(r)}) \\ A+2,\ C+2 : (B', D'); \ldots; (B^{(r)}, D^{(r)}) \end{matrix}$$

$$\left(\begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}], (-\eta : h_1, \ldots, h_r), (\alpha-\eta : h_1, \ldots, h_r) : \\ [(c) : C', \ldots, C^{(r)}], (-\alpha-\eta-R-1 : h_1, \ldots, h_r), (\alpha-\eta+R : h_1, \ldots, h_r) : \end{matrix} \right.$$

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$$\left[(b') : \phi' \right] ; \ldots ; \left[(b^{(r)}) : \phi^{(r)} \right] ; \begin{matrix} z_1 \, 2^{h_1} \\ \vdots \\ z_r \, 2^{h_r} \end{matrix} \Bigg) \qquad \ldots (2.2)$$
$$\left[(d') : \delta' \right] ; \ldots ; \left[(d^{(r)}) : \delta^{(r)} \right] ;$$

where,

$\mathrm{Re}(\eta) > -1$, $\mathrm{Re}(\alpha) > -1$, $h_i > 0, i = 1, \ldots, r$,

$|\arg(z_i)| < \frac{1}{2} T_i \pi$, $T_i > 0$ and

$$\mathrm{Re}\left(\eta + \sum_{i=1}^{r} h_i \frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1, i = 1, \ldots, r$$

$$\int_{-1}^{1} \frac{1}{\rho} (1 - t + \rho)^{-\alpha} (1 + t + \rho)^{-\beta} (1+x)^{\eta} (1-x)^{\alpha} H \begin{pmatrix} z_1 (1+x)^{h_1} \\ \vdots \\ z_r (1+x)^{h_r} \end{pmatrix} dx$$

$$= 2^{\eta-\beta+1} \sum_{R=0}^{\infty} \frac{\Gamma(\alpha+R+1)}{R!} t^R \, H \begin{matrix} 0, \varepsilon+2 : (M', N') ; \ldots ; (M^{(r)}, N^{(r)}) \\ A+2, C+2 ; (B', D') ; \ldots ; (B^{(r)}, D^{(r)}) \end{matrix}$$

$$\left(\begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}], (-\eta : h_1, \ldots, h_r), (\beta - \eta ; h_1, \ldots, h_r) ; \\ [(c) : C', \ldots, C^{(r)}], (-\alpha-\eta-R-1 ; e_1, \ldots, h_r), (\beta-\eta+R : h_1, \ldots, h_r) ; \end{matrix} \right.$$

$$\left[(b') : \phi' \right] ; \ldots ; \left[(b^{(r)}) : \phi^{(r)} \right] ; \begin{matrix} z_1 \, 2^{h_1} \\ \vdots \\ z_r \, 2^{h_r} \end{matrix} \Bigg) \qquad \ldots (2.3)$$
$$\left[(d') : \delta' \right] ; \ldots ; \left[(d^{(r)}) : \delta^{(r)} \right] ;$$

where

$Re(\eta) > -1$, $Re(\alpha) > -1$, $h_i > 0$,

$|\arg(z_i)| < \frac{1}{2} T_i \pi$, $T_i > 0$ and

$$Re\left(\eta + \sum_{i=1}^{r} h_i \, d_j^{(i)} / \delta_j^{(i)}\right) > -1, \quad i = 1, \ldots, r$$

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Next for the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ [11, *p.* 254, Eq.(1)], we have

$$P_k^{(\alpha,\beta)}(t+\rho)\,P_k^{(\alpha,\beta)}(t-\rho) = \frac{(-1)^k(1+\alpha)_k(1+\beta)_k}{(k!)^2} \cdot \sum_{r=0}^{k} \frac{(-k)_r(1+\alpha+\beta+k)_r}{(1+\alpha)_r(1+\beta)_r} P_r^{(\alpha,\beta)}(x)\,t^r \qquad \ldots(1.7)$$

$$\rho^k P_k^{(\alpha,\alpha)}\left(\frac{1-xt}{\rho}\right) = \frac{(1+\alpha)_k}{k!} \sum_{r=0}^{k} \frac{(-k)_r}{(1+\alpha)_r} P_r^{(\alpha,\alpha)}(x)\,t^r \qquad \ldots(1.8)$$

$$\frac{1}{\rho}(1-t+\rho)^{-\alpha}(1+t+\rho)^{-\beta} = 2^{-\alpha-\beta} \sum_{r=0}^{\infty} P_r^{(\alpha,\beta)}(x)\,t^r \qquad \ldots(1.9)$$

In each of the formulas (1.7), (1.8), and (1.9), and throughout this paper, $\rho = (1-2xt+t^2)^{1/2}$. Formulas (1.7), (1.7) and (1.9) can be found, for example in [4, p. 945], [4, p. 946] and [6, p. 172], respectively.

$$\int_{-1}^{1}(1+x)^{\eta}(1-x)^{\alpha}P_n^{(\alpha,\beta)}(x)H\begin{pmatrix} z_1(1+x)^{h_1} \\ \vdots \\ z_r(1+x)^{h_r} \end{pmatrix}dx$$

$$= \frac{2^{\eta+\alpha+1}\Gamma(\alpha+n+1)}{n!} H\begin{matrix} 0,\ \varepsilon+2 : (M', N'); \ldots; (M^{(r)}, N^{(r)}) \\ A+2,\ C+2 : (B', D'); \ldots; (B^{(r)}, D^{(r)}) \end{matrix}$$

$$\left(\begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}],\ (-\eta : h_1, \ldots, h_r),\ (\beta-\eta : h_1, \ldots, h_r) : \\ [(c) : C', \ldots, C^{(r)}],\ (-\alpha-\eta-n-1 : h_1, \ldots, h_r),\ (\beta-\eta+n : h_1, \ldots, h_r) : \end{matrix}\right.$$

$$\left.\begin{matrix} [(b') : \phi'];\ldots;[(b^{(r)}) : \phi^{(r)}];\ 2^{h_1}z_1 \\ [(d') : \delta'];\ldots;[(d^{(r)}) : \delta^{(r)}];\ 2^{h_r}z_r \end{matrix}\right) \qquad \ldots(1.10)$$

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Formula (1. 10) was given by Srivastava and Panda [13, p. 131, Eq. (2.2)] .

$$\int_{-1}^{1} (1-x)^{\eta} (1+x)^{\sigma} P_n^{(\alpha,\beta)}(x) H\begin{pmatrix} z_1 (1-x)^{h_1} (1+x)^{k_1} \\ z_r (1-x)^{h_r} (1+x)^{k_r} \end{pmatrix} dx$$

$$= 2^{\eta+\sigma+1} \sum_{m=0}^{n} \frac{(-n)_m (\alpha+\beta+n+1)_m}{(\alpha+1)_m \, m!}$$

$$H \begin{matrix} 0, \varepsilon+2 : (M', N') ; \ldots ; (M^{(r)}, N^{(r)}) \\ A+2, C+1 : (B', D') ; \ldots ; (B^{(r)}, D^{(r)}) \end{matrix}$$

$$\left(\begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}], (-\sigma : k_1, \ldots, k_r), (-m-\eta : h_1, \ldots, h_r) : \\ [(c) : C', \ldots, C^{(r)}], (-1-m-\eta-\sigma : h_1+k_1, \ldots, h_r+k_r) : \end{matrix} \right.$$

$$\left. \begin{matrix} [(b') : \phi'] ; \ldots ; [(b^{(r)}) : \phi^{(r)}] ; z_1 2^{h_1+k_1} \\ [(d') : \delta'] ; \ldots ; [(d^{(r)}) : \delta^{(r)}] ; z_r 2^{h_r+k_r} \end{matrix} \right) \quad \ldots (1.11)$$

The integral (1.11) can be evaluated by making use of a known result [7, p. 284, Fq. (2)] . In fact, (1.12) is an obvious special case of much more general results given by Srivastava and Singh [15, p. 172, Fq. (3.2)] .

2. MAIN INTEGRLS

The following integrals have been evaluated in this paper :

$$\int_{-1}^{1} (1+x)^{\eta} (1-x)^{\alpha} P_k^{(\alpha,\beta)}(t+\rho) P_k^{(\alpha,\beta)}(t-\rho) H\begin{pmatrix} z_1 (1+x)^{h_1} \\ \cdot \\ z_r (1+x)^{h_r} \end{pmatrix} dx$$

$$= 2^{\eta+\alpha+1} \frac{(-1)^k \Gamma(1+\alpha+k)(1+\beta)_k}{(k!)^2}$$

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$$\int_{-1}^{1} (1-x)^{\eta} (1+x)^{\sigma} P_k^{(\alpha,\beta)}(t+\rho) P_k^{(\alpha,\beta)}(t-\rho) H\begin{pmatrix} z_1 (1-x)^{h_1} (1+x)^{k_1} \\ \vdots \\ z_r (1-x)^{h_r} (1+x)^{k_r} \end{pmatrix} dx$$

$$= 2^{\eta+\sigma+1} \frac{(-1)^k \Gamma(1+\alpha+k) \Gamma(1+\beta+k)}{(k!)^2}$$

$$\sum_{R=0}^{k} \sum_{m=0}^{R} \frac{(-k)_r (1+\alpha+\beta+k)_R}{\Gamma(1+\alpha+R) \Gamma(1+\beta+R)} t^R \frac{(-R)_m (\alpha+\beta+R+1)_m}{(\alpha+1)_m \, m!}$$

$$H \begin{matrix} 0, \epsilon+2 : (M', N') ; \quad . ; (M^{(r)}, N^{(r)}) \\ A+2, C+1 : (B', D) ; \quad . ; (B^{(r)}, D^{(r)}) \end{matrix}$$

$$\left(\begin{matrix} [(a) : \theta', \ldots \theta^{(r)}], (-\sigma : k_1, \ldots, k_r), (m-\eta_i : h_1, \ldots, h_r) : \\ [(c) : C', \ldots, C^{(r)}], (-1-m-\eta_i-\sigma : h_1+k_1, \ldots, h_r+k_r) : \end{matrix} \right.$$

$$\left. \begin{matrix} [(b) : \phi'] ; \ldots ; [(b^{(r)}) : \phi^{(r)}] ; z_1 2^{h_1+k_1} \\ [(d') : \delta'] ; \ldots ; [(d^{(r)}) : \delta^{(r)}] ; z_r 2^{h_r+k_r} \end{matrix} \right) \quad \ldots (2.4)$$

where

$Re(\eta_i) < -1$, $Re(\sigma) < -1$, $k_i > 0$,

$h_i > 0$, $|arg(z_i)| < \frac{1}{2} T_i \pi$ and

$$Re\left(\eta_i + \sum_{i=1}^{r} h_i d_j^{(i)} / \delta_j^{(i)}\right) > -1, \quad Re\left(\sigma + \sum_{i=1}^{r} k_i d_j^{(i)} / \delta_j^{(i)}\right) > -1,$$

$T_i > 0$, $i = 1, \ldots, r$

$$\int_{-1}^{1} (1-x)^{\eta} (1+x^{\sigma} \rho^k P_k^{(\alpha,\alpha)} \left(\frac{1-xt}{\rho}\right) H\begin{pmatrix} z_1 (1-x)^{h_1} (1+x)^{k_1} \\ \vdots \\ z_r (1-x)^{h_r} (1+x)^{k_r} \end{pmatrix} dx$$

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$$= \frac{2^{\eta+\alpha+1}\,\Gamma(1+\alpha+k)}{k!} \sum_{R=0}^{k} \sum_{m=0}^{r} \frac{(-k)_R}{\Gamma(1+\alpha+R)}\, t^R\, \frac{(-R)_m\,(2\alpha+R+1)_m}{(\alpha+1)_m\, m!}$$

$$0,\ \varepsilon+2 : (M', N')\ ;\ \ldots\ ;\ (M^{(r)}, N^{(r)})$$
$$.H$$
$$A+2,\ C+1 : (B', D')\ ;\ \ldots\ ;\ (B^{(r)}, D^{(r)})$$

$$\left(\begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}],\ (-\sigma : k_1, \ldots, k_r),\ (-m-\eta : h_1, \ldots, h_r) : \\ [(c) : C', \ldots, C^{(r)}],\ (-1-m-\gamma_i-\sigma : : h_1+k_1, \ldots, h_r+k_r) : \end{matrix}\right.$$

$$\left.\begin{matrix} [(b') : \phi']\ ;\ \ldots\ ;\ [(b^{(r)}) : \phi^{(r)}]\ ; \\ [(d') : \delta']\ ;\ \ldots\ ;\ [(d^{(r)}) : \delta^{(r)}]\ , \end{matrix} \begin{matrix} z_1\, 2^{h_1+k_1} \\ \vdots \\ z_r\, 2^{h_r+k_r} \end{matrix}\right) \qquad \ldots (2.5)$$

where

$$\mathrm{Re}\,(\eta) > -1,\ \mathrm{Re}\,(\sigma) > -1,\ h_i > 0,\ k_i > 0,$$

$$|\arg(z_i)| < \tfrac{1}{2} T_i \pi, \text{ and } \mathrm{Re}\left(\eta + \sum_{i=1}^{r} h_i\, d_j^{(i)} / \delta_j^{(i)}\right) > -1.$$

$$\mathrm{Re}\left(\sigma + \sum_{i=1}^{r} k_i\, d_j^{(i)}\right) > 1,\quad i = 1, \ldots, r$$

$$\int_{-1}^{1} (1-x)^{\eta} (1+x)^{\sigma} \frac{1}{\rho} (1-t+\rho)^{-\alpha} (1+t+\rho)^{-\beta}\, H\begin{pmatrix} z_1 (1-x)^{h_1} (1+x)^{k_1} \\ z_r (1-x)^{h_r} (1+x)^{k_r} \end{pmatrix} dx$$

$$= 2^{-\alpha-\beta+\eta+\sigma+1} \sum_{R=0}^{\infty} \sum_{m=0}^{R} t^R\, \frac{(-R)_m\,(\alpha+\beta+R+1)_m}{(\alpha+1)_m\, m!}$$

$$0,\ \varepsilon+2 : (M', N')\ ;\ \ldots\ ;\ (M^{(r)}, N^{(r)})$$
$$. H$$
$$A+2,\ C+1 : (B', D')\ ;\ \ldots,\ (B^{(r)}, D^{(r)})$$

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$$\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}],(-\sigma:k_1,\ldots,k_r),(-m-\eta:h_1,\ldots,h_r):\\ [(c):C',\ldots,C^{(r)}],(-1-m-\eta-\sigma:h_1+k_1,\ldots,h_r+k_r):\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];z_1\,2^{h_1+k_1}\\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];z_r\,2^{h_r+k_r}\end{matrix}\right) \quad \ldots (2.6)$$

where

$Re\,(\eta) > -1$, $Re\,(\sigma) > -1$, $h_i > 0$, $k_i > 0$,

$|\arg(z_i)| < \frac{1}{2} T_i \pi$, > 0 and

$$Re\left(\eta + \sum_{i=1}^{r} h_i d_j^{(i)} / \delta_j^{(i)}\right) > -1,$$

$$Re\left(\sigma + \sum_{i=1}^{r} k_i d_j^{(i)} / \delta_j^{(i)}\right) > -1,\ i = , \ldots, r$$

Proof : To prove (2.1), multiply both the sides of (1.7) by

$(1+x)^{\eta}(1-x)^{\alpha} H\begin{pmatrix} z_1 (1+x)^{h_1} \\ \vdots \\ z_r (1+x)^{h_r}\end{pmatrix}$ and integrating both sides with respect to x between the iimits -1 to 1, we get

$$\int_{-1}^{1} (1+x)^{\eta}(1-x)^{\alpha} P_k^{(\alpha,\beta)}(t+\rho) P_k^{(\alpha,\beta)}(t-\rho) H\begin{pmatrix} z_1 (1+x)^{h_1} \\ \vdots \\ z_r (1+x)^{h_r}\end{pmatrix} dx$$

$$= \int_{-1}^{1} \frac{(-1)^k (1+\alpha)_k (1+\beta)_k}{(k!)^2} \sum_{R=0}^{k} \frac{(-k)_R}{(1+\alpha)_R} \frac{(1+\alpha+\beta+k)_R}{(1+\beta)_R}$$

$$. P_k^{(\alpha,\beta)}(x)\, t^R (1+x)^{\eta}(1-x)^{\alpha} H\begin{pmatrix} z_1 (1+x)^{h_1} \\ \vdots \\ z_r (1+x)^{h_r}\end{pmatrix} dx \quad \ldots (2.7)$$

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Now, on interchanging the order of integration and summation in the right-hand side of (2.7), which is permissible as the series involved is finite and integrals involved are absolutely convergent and on evaluating the x-integral thus obtained with the help of (1.10), we get the desired result under the conditions mentioned therein .

The proofs of the formulas (2.2) to (2,6) can be peveloped by proceeding on similar lines with the help of the result (1.8) and (1.10), (1.9). and (1.10), (1.9) and (1.11), (1.8) and (1.11), (1.9) and (1.11), respectively.

Acknowledgements

The author is grateful to Prof. H. M. Stivastava (University of Victoria, Canada) for giving suggestions for revising of the paper.

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Jñānābha, Vol. 18, 1988

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ABSTRACT

The object of the present paper is to derive some expansion formulae for the multivariable H-function involving exponential functions, Legendre functions and Bessel functions .

1. INTRODUCTION

Srivastava and Panda [10] have defined the multivariable H-function as (see also [9] and [11])

$$H \begin{matrix} 0, \varepsilon : (u', v') ; \ldots ; (u^{(r)}, v^{(r)}) \\ A, C : (B', D') ; \ldots ; (B^{(r)}, D^{(r)}) \end{matrix}$$

$$\left[\begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}] : [(b') : \phi'] , \ldots , [(b^{(r)}) : \phi^{(r)}] ; \\ [(c) : \Psi', \ldots, \Psi^{(r)}] : [(d') : \delta'] ; \ldots ; [(d^{(r)}) : \delta^{(r)}] ; \end{matrix} z_1 , \ldots , z_r \right]$$

$$= \frac{1}{(2\pi i)^r} \int_{L_1} \cdots \int_{L_r} U_1(s_1) \ldots U_r(s_r) \, V(s_1, \ldots, s_r)$$

$$z_1^{s_1} \ldots z_r^{s_r} \, ds_1 \ldots ds_r \qquad \ldots (1.1)$$

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$$U_i(s_i) = \frac{\prod_{j=1}^{u^{(i)}} \Gamma(d_j^{(i)} - \delta_j^{(i)} s_i) \prod_{j=1}^{v^{(i)}} \Gamma(1 - b_j^{(i)} + \phi_j^{(i)} s_i)}{\prod_{j=1+u^{(i)}}^{D^{(i)}} \Gamma(1 - d_j^{(i)} + \delta_j^{(i)} s_i) \prod_{j=v^{(i)}+1}^{B^{(i)}} \Gamma(b_j^{(i)} - \phi_j^{(i)} s_i)}$$

$$i = 1, \ldots, r \qquad \ldots (1.2)$$

and

$$V(s_1, \ldots, s_r) = \frac{\prod_{j=1}^{\epsilon} \Gamma(1 - a_j + \sum_{i=1}^{r} \theta_j^{(i)} s_i)}{\prod_{j=\epsilon+1}^{A} \Gamma(a_j - \sum_{i=1}^{r} \theta_j^{(i)} s_i) \prod_{j=1}^{C} \Gamma(1 - c_j + \sum_{i=1}^{r} \Psi_j^{(i)} s_i)}$$

$$\ldots (1.3)$$

and, for convergence,

$$|\arg(z_i)| < \tfrac{1}{2} T_i \pi, \quad i = 1, \ldots, r \qquad \ldots (1.4)$$

where,

$$T_i = -\sum_{j=\epsilon+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{v^{(i)}} \theta_j^{(i)} - \sum_{j=v^{(i)}+1}^{B^{(i)}} \theta_j^{(i)} - \sum_{j=1}^{C} \Psi_j^{(i)}$$

$$+ \sum_{j=1}^{u^{(i)}} \delta_j^{(i)} - \sum_{j=u^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} > 0, \quad i = 1, \ldots, r \qquad \ldots (1.5)$$

2. PRELIMINARY RESULTS

The following known results [7, p. 340 (95) ; 4, p. 316 ; 8, p. 78; 3, p. 62 ; 12, p. 324 ; 6, p. 291 (6)] will be required in the proof of the integrals and expansions:

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$$\int_{-\pi/2}^{\pi/2} (\cos\theta)^{m+n-2}\, e^{i(m-n)\theta}\, d\theta = \frac{\pi\Gamma(m+n-1)}{2^{m+n-2}\,\Gamma(m)\,\Gamma(n)}\,,\quad Re\,(m+n) > 1 \qquad \ldots (2.1)$$

$$\int_{-1}^{1} (1-x^2)^{\alpha-1}\, P_\beta^\delta(x)\,dx = \frac{2^\delta\pi\,\Gamma(\alpha+\delta/2)\,\Gamma(\alpha-\delta/2)}{\Gamma(\alpha+\frac{\beta}{2}+1)\,\Gamma(\alpha-\frac{\beta}{2})\,\Gamma(\frac{2-\delta+\beta}{2})\,\Gamma(\frac{1-\beta-\delta}{2})} \qquad \ldots (2.2)$$

$\mathrm{Re}\,(2\alpha+\delta) > 0,\ \mathrm{Re}\,(2\alpha-\delta) > 0.$

$$\int_0^\infty e^{ix}\, x^\alpha\, J_\beta(x)\,dx = \frac{e^{1/2(i(\alpha+\beta+1)\pi)}\,\Gamma(\alpha+\beta+1)\,\Gamma(-\alpha-\frac{1}{2})}{2^{\alpha+1}\,\Gamma(\frac{1}{2})\,\Gamma(\beta-\alpha)} \qquad \ldots (2.3)$$

$\mathrm{Re}\,(\alpha+\beta) > -1,\ Re\,(\alpha) < -\frac{1}{2}.$

$$\int_0^\infty t^{s-1}\, J_\beta(\alpha t)\, J_\delta(\alpha t)\,dt = \frac{2^{s-1}\,\alpha^{-s}\,B(1-s,\frac{\beta+\delta+s}{2})}{\Gamma(\frac{\delta-\beta-s}{2}+1)\,\Gamma(\frac{\beta-\delta-s}{2}+1)} \qquad \ldots (2.4)$$

$-Re\,(\beta+\delta) < Re\,(s) < 1,\ \alpha > 0.$

$$\int_a^b e^{\frac{2m\pi ix}{a-b}}\, e^{\frac{2n\pi ix}{a-b}}\,dx = 0,\ if\ m \neq -n \qquad \ldots (2.5)$$

$$= b-a,\ if\ m = -n$$

$$\int_{-1}^{1} P_n^m(x)\, P_r^m(x)\,dx = 0,\ if\ r \neq n$$

$$= \frac{2}{2n+1}\cdot\frac{(n+m)!}{(n-m)!}\quad if\ r = n \qquad \ldots (2.6)$$

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$$\int_0^\infty t^{-1} J_{\alpha+2n+1}(t)\, J_{\alpha+2m+1}(t)\, dt = 0, \quad \text{if } m \neq n$$

$$= (4n+2\alpha+2)^{-1}, \text{ if } m = n \qquad \ldots (2.7)$$

$Re(a) + m + n > -1$.

3. MAIN INTEGRALS

$$\int_{-\pi/2}^{\pi/2} (\cos\theta)^{m+n-2}\, e^{i(m-n)\theta}\, H \begin{matrix} 0, \varepsilon : (u', v') ; \ldots ; (u^{(r)}, v^{(r)}) \\ A, C : (B', D') ; \ldots ; (B^{(r)}, D^{(r)}) \end{matrix}$$

$$\left[\begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}] : [(b') : \phi'] ; \ldots ; [(b^{(r)}) : \phi^{(r)}] ; \\ [(c) : \Psi', \ldots, \Psi^{(r)}] : [(d') : \delta'] ; ; \ldots ; [(d^{(r)}) : \delta^{(r)}] ; \end{matrix} (e^{i\theta}\cos\theta)^{h_1/2}\, z_1, z_2, \ldots, z_r \right] d\theta$$

$$= \frac{2}{2^{m+n}\,\Gamma(n)} \sum_{i,-i} \frac{1}{i} H \begin{matrix} 0, \varepsilon : (u'+1, v'+1) ; (u'', v'') ; \\ A, C : (B'+2, D'+2) ; (B'', D'') ; \end{matrix}$$

$$\ldots ; (u^{(r)}, v^{(r)}) \quad \left[\begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}] : [(b') : \phi'], (1, h_1), \\ [(c) : \Psi', \ldots, \Psi^{(r)}] : [(d') : \delta'], (1, h_1), \end{matrix}\right.$$

$$\ldots ; (B^{(r)}, D^{(r)}) \quad \begin{matrix} (2-m-n, \frac{h_1}{2}) ; [(b'') : \phi''] ; \ldots ; [(b^{(r)}) : \delta^{(r)}] ; \\ (1-m, \frac{h_1}{2}) ; [(d'') : \phi''] ; \ldots ; [(d^{(r)}) : \delta^{(r)}] ; \end{matrix}$$

$$\left. \frac{e^{i\pi h_1}}{2^{h_1/2}}\, z_1, z_2, \ldots, z_r \right] \qquad \ldots (3.1)$$

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where h_1 is a positive number and $Re\,(m+n) > 1$ and conditions of (1.1) are also satisfied. The symbol $\sum_{i,-i}$ means that in the expression following it, i is to be replaced by $-i$ and the two expressions so obtained must be added.

$$\int_{-1}^{1} (1-x^2)^{\alpha-1} P_{\beta}^{\delta}(x)\, H\begin{matrix} 0, \varepsilon : (u', v') ; \ldots ; (u^{(r)}, v^{(r)}) \\ A, C : (B', D') ; \ldots ; (B^{(r)}, D^{(r)}) \end{matrix}$$

$$\left[\begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}] : [(b') : \phi'] ; \ldots ; [(b^{(r)}) : \phi^{(r)}] ; \\ [(c) : \Psi', \ldots, \Psi^{(r)}] : [(d') : \delta'] ; \ldots ; [(d^{(r)}) : \delta^{(r)}] ; \end{matrix}\right.$$

$$\left. z_1 (1-x^2)^{h_1}, \ldots, z_r (1-x^2)^{h_r} \right] dx$$

$$= \frac{2\delta\pi}{\Gamma\left(\frac{2-\delta+\beta}{2}\right)\Gamma\left(\frac{1-\delta-\beta}{2}\right)} H\begin{matrix} 0, \varepsilon+2 : (u', v') ; \ldots ; (u^{(r)}, v^{(r)}) \\ A+2, C+2 : (B', D') ; \ldots ; (B^{(r)}, D^{(r)}) \end{matrix}$$

$$\left[\begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}], \left(1-\alpha-\frac{\delta}{2} ; h_1, \ldots, h_r\right), \left(1-\alpha+\frac{\delta}{2} ; h_1, \ldots, h_r\right) : \\ [(c) : \Psi', \ldots, \Psi^{(r)}], \left(-\alpha-\frac{\beta}{2} ; h_1, \ldots, h_r\right), \left(1-\alpha+\frac{\beta}{2} ; h_1, \ldots, h_r\right) : \end{matrix}\right.$$

$$\left.\begin{matrix} [(b') : \phi'] ; \ldots ; [(b^{(r)}) : \phi^{(r)}] ; \\ [(d') : \delta'] ; \ldots, [(d^{(r)}) : \delta^{(r)}] ; \end{matrix} z_1, \ldots, z_r \right] \qquad \ldots (3.2)$$

where

$$|\arg(z_i)| < \tfrac{1}{2} T_i \pi,\ T_i > 0, \text{ and}$$

$$2\,Re\left(\alpha + \sum_{i=1}^{r} h_i \frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > |\,\mathrm{Re}\,(\delta)\,|,\ j = 1, \ldots, u^{(i)}.$$

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$$\int_0^\infty e^{ix} x^\alpha J_\beta(x)\, H\begin{matrix}0,\ \varepsilon : (u', v') ; \ldots ; (u^{(r)}, v^{(r)}) \\ A,C : (B'. D') ; \ldots ; (B^{(r)}, D^{(r)})\end{matrix}\left[\begin{matrix}[(a) : \theta', \ldots, \theta^{(r)}]: \\ [(c) : \Psi', \ldots, \Psi^{(r)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b') : \phi'] ; \ldots ; [(b^{(r)}) : \phi^{(r)}] ; \\ [(d') : \delta'] ; \ldots ; [(d^{(r)}) : \delta^{(r)} ;\end{matrix}\ x^{h_1} z_1, z_2, \ldots, z_r\right] dx$$

$$\frac{e^{\frac{1}{2}i(\alpha+\beta+1)\pi}}{\Gamma(\frac{1}{2})\, 2^{\alpha+1}} H\begin{matrix}0,\ \varepsilon : (u'+1, v'+1) ; (u'', v'') ; \ldots ; (u^{(r)}, v^{(r)}) \\ A,C : B'+2, D'+1) ; (B'', D'') ; \ldots ; (B^{(r)}, D^{(r)})\end{matrix}$$

$$\left[\begin{matrix}[(a) : \theta', \ldots, \theta^{(r)}] : [(b') : \phi'], (-\alpha-\beta, h_1), (\beta-a, h_1); [(b'') : \phi''] ; \\ [(c) : \Psi', \ldots, \Psi^{(r)}]: [(d') : \delta'], (-\alpha-\frac{1}{2}, h_1) \quad ; [(d'') : \delta'']\end{matrix}\right.$$

$$\left.\begin{matrix}\ldots ; [(b^{(r)}) : \phi^{(r)}] ; \\ \ldots ; [(d^{(r)}) : \delta^{(r)}] ;\end{matrix}\ \left(\frac{e^{\frac{1}{2}i\pi}}{2}\right) h_1\ z_1, z_2, \ldots, z_r\right]$$

... (3.3)

where $|arg(z_i)| < \frac{1}{2} T_i \pi$, $T_i > 0$, and

$$Re\left(\alpha+\beta+h_1 \frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1,\ Re(\alpha) < \tfrac{1}{2},\ j=1,2,\ldots,u^{(i)}.$$

$$\int_0^\infty x^{-k} J_\delta(ax) J_\beta(\alpha x)\, H\begin{matrix}0,\ \varepsilon : (u', v') ; \ldots ; (^{(r)}, v^{(r)}) \\ A, C : (B', D') ; \ldots ; (B^{(r)}, D^{(r)})\end{matrix}$$

$$\left[\begin{matrix}[(a) : \theta', \ldots, \theta^{(r)}] : [(b') : \phi'] ; \ldots ; [(b^{(r)}) : \phi^{(r)}] ; \\ [(c) : \Psi', \ldots, \Psi^{(r)}] : [(d') : \delta'] ; \ldots ; (d^{(r)}) ; \delta^{(r)}] ;\end{matrix}\right.$$

$$\left.z_1 x^{2h_1}, z_2, \ldots, z_r\right] dx$$

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$$= \frac{1}{2\sqrt{\pi}}\, a^{k-1}\, H^{0,\varepsilon:(u'+2,\,v'+1);(u'',\,v'');\ldots;(u^{(r)},\,v^{(r)})}_{A,\,C:(B'+4,\,D'+2);(B'',\,D'');\ldots;(B^{(r)},\,D^{(r)})}\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\\ [(c):\Psi',\ldots,\Psi^{(r)}]:[(d'):\delta'];\end{matrix}\right.$$

$$\left(\frac{\beta}{2}-\frac{\delta}{2}+\frac{k}{2}+\frac{1}{2},\,h_1\right),$$

$$\left(\frac{k}{2},\,h_1\right),\left(\frac{k}{2}+\frac{1}{2},\,h_1\right)$$

$$\left(\frac{1}{2}+\frac{k}{2}-\frac{\beta}{2}-\frac{\delta}{2},\,h_1\right),\left(\frac{1}{2}+\frac{\delta}{2}+\frac{\beta}{2}+\frac{k}{2},\,h_1\right),$$

$$\left(\frac{\delta}{2}-\frac{\beta}{2}+\frac{k}{2}+\frac{1}{2},\,h_1\right);[(b''):\phi'']\,;\ldots;[(b^{(r)}):\phi^{(r)}]\,;$$

$$\text{---------- }\,;[(d''):\phi'']\,;\ldots;[(d^{(r)}):\delta^{(r)}]\,;$$

$$\left.z_1\,a^{-2h_1},\,z_2,\ldots,z_r\right] \qquad \ldots(3.4)$$

where $|\arg(z_i)| < \frac{1}{2}T_i\pi,\ T_i > 0$, and

$$Re\left(\beta+\delta-k+2h_1\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1,$$

$$Re\left(2h_1\frac{d_j^{(i)}}{\delta_j^{(i)}}-k\right) < 2h_1,\ j=1\ldots,u^{(i)}$$

Proof : Equation (3.1) can be proved on using the contour integral (1.1), interchanging the order of integration, which is justified under

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the conditions stated with the result, evaluating the snner integral with the help of (2.1) and using a relation $\Gamma(\alpha)\,\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha}$

$= \frac{2\pi i}{e^{i\pi\alpha} - e^{-i\pi\alpha}}$, and applying the definition (1 1) .

The proof of the formula (3.2) to (3.4) can be developed by proceeding on similar lines with the help of the result (2.2) to (2.4) respectively in place of result (2.1) and a relation $\Gamma(\alpha)\,\Gamma(1-\alpha) = -\frac{\pi}{\sin \pi\alpha}$.

4. EXPANSION FORMULAE

$$(\cos\theta)^{2n-2}\, H\begin{matrix} 0, \varepsilon : (u', v') ; \ldots ; (u^{(r)}, v^{(r)}) \\ A, C : (B', D') ; \ldots ; (B^{(r)}, D^{(r)}) \end{matrix}\left[\begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}] : \\ [(c) : \Psi, \ldots, \Psi^{(r)}] : \end{matrix}\right.$$

$$\left.\begin{matrix} [(b') : \phi'] ; \ldots ; [(b^{(r)}) : \phi^{(r)}] ; \\ [(d') : \delta'] ; \ldots ; [(d^{(r)}) : \delta^{(r)}] ; \end{matrix}\; (e^{i\theta}\cos\theta)^{h_1/2}\; z_1, z_2, \ldots, z_r\right]$$

$$= 2\left\{\sum_{R=-\infty}^{\infty} \frac{1}{2^{-R}\Gamma(R)} \sum_{i,-i} \frac{1}{i}\right.$$

$$H\begin{matrix} 0, \varepsilon : (u'+1, v'+2) ; (u'', v'') ; \ldots ; (u^{(r)}, v^{(r)}) \\ A, C : B'+2, D'+2) ; (B'', D'') ; \ldots ; (B^{(r)}, D^{(r)}) \end{matrix}$$

$$\left[\begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}] : [(b') : \phi'], (1, h_1), (2-2R, \frac{h_1}{2}) ; [(b'') : \phi'] ; \\ [(c) : \Psi', \ldots, \Psi^{(r)}] : [(d') : \delta'], (1, h_1), (1-R, \frac{h_1}{2}) ; [(d'') : \delta''] ; \end{matrix}\right.$$

$$\left.\left.\begin{matrix} \ldots ; [(b^{(r)}) : \phi^{(r)}] ; \\ \ldots ; [(d^{(r)}) : \delta^{(r)}] ; \end{matrix}\; \frac{e^{i\pi h_1}}{2^{h_1/2}}\; z_1, z_2, \ldots, z_r\right]\right\} e^{-2i(R-n)\theta}$$

... (4.1)

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where h_1 is a positive number, $-\pi/2 < \theta < \pi/2$, and $Re\,(n) > 1/2$.

$$(1-x^2)^{\alpha-1} \; H \begin{matrix} 0, \varepsilon : (u', v') ; \ldots ; (u^{(r)}, v^{(r)}) \\ A, C : (B', D') ; \ldots ; (B^{(r)}, D^{(r)}) \end{matrix}$$

$$\left[\begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}] : [(b') : \phi'] ; \ldots ; [(b^{(r)}) : \phi^{(r)}] ; \\ [(c) : \Psi', \ldots, \Psi^{(r)}] : [(d') : \delta'] ; \ldots ; [(d^{(r)}) : \delta^{(r)}] ; \end{matrix} \right.$$

$$\left. z_1 (1-x^2)^{h_1}, \ldots, z_r (1-x^2)^{h_r} \right]$$

$$= 2^{\delta-1} \pi \sum_{R=0}^{\infty} \frac{(2R+1)(R-\delta)!}{(R+\delta)! \, \Gamma\left(\frac{2-\delta+R}{2}\right) \Gamma\left(\frac{1-\delta-R}{2}\right)}$$

$$H \begin{matrix} 0, \varepsilon+2 : (u', v') ; \ldots : (u^{(r)}, v^{(r)}) \\ A+2, C+2 : (B', D') ; \ldots ; (B^{(r)} D^{(r)}) \end{matrix}$$

$$\left[\begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}], (1-\alpha-\frac{\delta}{2} ; h_1, \ldots, h_r), (-\alpha+\frac{\delta}{2} ; h_1, \ldots, h_r) ; \\ [(c) : \Psi', \ldots, \Psi^{(r)}], (-\alpha-\frac{R}{2} ; h_1, \ldots, h_r), (1-\alpha+\frac{R}{2} ; h_1, \ldots, h_r) ; \end{matrix} \right.$$

$$\left. \begin{matrix} [(b') : \phi'] ; \ldots ; [(b^{(r)}) : \phi^{(r)}] ; \\ [(d') : \delta'] ; \ldots, [(d^{(r)}) : \delta^{(r)}] ; \end{matrix} \; z_1, \ldots, z_r \right] P_R^{\delta}(x) \qquad \ldots (4.2)$$

where

$$|\arg(z_i)| < \tfrac{1}{2} T_i \pi, \; T_i > 0, \text{ and}$$

$$Re\left(\alpha + \sum_{i=1}^{r} h_i \frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > |\mathrm{Re}\,(\delta)|, \; j = 1, \ldots, u^{(i)} ;$$

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$$x^{\alpha}\ H\begin{matrix}0,\ \varepsilon : (u', v') ; \ldots ; (u^{(r)}, v^{(r)}) \\ A, C : (B', D') ; \ldots ; (B^{(r)}, D^{(r)})\end{matrix}\left[\begin{matrix}[(a) : \theta', \ldots, \theta^{(r)}] \\ [(c) : \Psi', \ldots, \Psi^{(r)}]\end{matrix}\right.$$

$$\left.\begin{matrix}[(b') : \phi'] ; \ldots ; [(b^{(r)}) : \phi^{(r)}] ; \\ [(d') : \delta'] ; \ldots ; [(d^{(r)}) ; \delta^{(r)}] ;\end{matrix}\quad x^{h_1}\ z_1, z_2, \ldots, z_r\right]$$

$$= \frac{1}{\Gamma(\frac{1}{2})2^{\alpha-1}} \sum_{R=0}^{\infty} e^{\frac{1}{2}i(\alpha+\gamma_i)\pi}$$

$$\eta\ H\begin{matrix}0,\ \varepsilon : (u'+1, v'+1) ; (u'', v'') ; \ldots ; (u^{(r)}, v^{(r)}) \\ A, C : B'+2, D'+1) ; (B'', D'') ; \ldots ; (B^{(r)}, D^{(r)})\end{matrix}$$

$$\left[\begin{matrix}[(a) : \theta', \ldots, \theta^{(r)}] : [(b') : \phi'], (-\alpha+1-\gamma_i h_1), \\ [(c) : \Psi', \ldots, \Psi^{(r)}] : [(d') : \delta'], (-\alpha+\frac{1}{2}, h_1)\end{matrix}\right.$$

$$(\gamma_i-\alpha+1 ; h_1) ; [(b'') : \phi''] ; \ldots ; [(b^{(r)}) : \phi^{(r)}] ;$$

$$; [(d'') : \delta''] ; \ldots ; [(d^{(r)}) : \delta^{(r)}] ;$$

$$\left.\left(\frac{e^{\frac{1}{2}i\pi}}{2}\right)^{h_1}\ z_1, z_2, \ldots, z_r\right] J_n(x) \qquad \ldots (4.3)$$

where $\gamma_i = n + 2R + 1$,

$|\arg(z_i)| < \frac{1}{2} T_i \pi$, $T_i > 0$, and

$$Re\left(\alpha + h_1 \frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0,\ Re\ \alpha < \frac{1}{2},\ j = 1, \ldots, u^{(i)}.$$

$$x^{-k} J_{\beta}(\alpha x)\ H\begin{matrix}0,\ \varepsilon : (u', v') ; \ldots ; (u^{(r)}, v^{(r)}) \\ A, C : (B', D') ; \ldots ; (B^{(r)}, D^{(r)})\end{matrix}\left[\begin{matrix}[(a) : \theta', \ldots, \theta^{(r)}] : \\ [(c) : \Psi', \ldots, \Psi^{(r)}] :\end{matrix}\right.$$

$$\left.\begin{matrix}[(b') : \phi'] ; \ldots ; [(b^{(r)}) : \phi^{(r)}] ; \\ [(d') : \delta'] ; \ldots ; [(d^{(r)}) : \delta^{(r)}] ;\end{matrix}\quad z_1\ x^{2h_1}, z_2, \ldots, z_r\right]$$

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$$= \frac{a^k}{\sqrt{\pi}} \sum_{R=0}^{\infty} \eta \; H \begin{matrix} 0, \varepsilon : (u'+2, v'+1) ; (u'', v'') ; \ldots ; (u^{(r)}, v^{(r)}) \\ A, C : (B'+4, D'+2) ; (B'', D'') ; \ldots ; (B^{(r)}, D^{(r)}) \end{matrix}$$

$$\left[\begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}] : [(b') : \phi'], (\frac{\beta}{2} - \frac{\eta}{2} + \frac{k}{2} + 1, h_1), \\ [(c) : \Psi', \ldots, \Psi^{(r)}] : [(d') : \delta'], (\frac{k}{2} + \frac{1}{2}, h_1) \end{matrix} \right.$$

$$(1 + \tfrac{k}{2} - \tfrac{\beta}{2} - \tfrac{\eta}{2}, h_1), (1 + \tfrac{\eta}{2} + \tfrac{\beta}{2} + \tfrac{k}{2}, h_1), (\tfrac{\eta}{2} - \tfrac{\beta}{2} + \tfrac{k}{2} + 1, h_1) ;$$

$$(\tfrac{k}{2} + 1, h_1) \qquad ;$$

$$\left. \begin{matrix} [(b'') : \phi''] ; \ldots ; [(b^{(r)}) : \phi^{(r)}] ; \\ [(d'') : \delta''] ; \ldots ; [(d^{(r)}) : \delta^{(r)}] ; \end{matrix} z_1 \alpha^{-2h_1}, z_2, \ldots, z_r \right] J_\eta(x), \quad \ldots (4.4)$$

where $\eta = u + 2R + 1$, $|arg(z_i)| < \frac{1}{2} T_i \pi$, $T_i > 0$, and

$$Re\left(\beta - k + 2h_1 \frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0,$$

$$Re\left(2h_1 \frac{b_j^{(i)}}{\phi_j^{(i)}} - k\right) < 2h_1 + 1, \; j = 1, \ldots, u^{(i)}.$$

Proof : To prove (4.1), let

$$f(\theta) = (\cos\theta)^{2n-2} \; H \begin{matrix} 0, \varepsilon : (u', v') ; \ldots ; (u^{(r)}, v^{(r)}) \\ A, C : (B', D') ; \ldots ; (B^{(r)}, D^{(r)}) \end{matrix} \left[\begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}] : \\ [(c) : \Psi', \ldots, \Psi^{(r)}] : \end{matrix} \right.$$

$$\left. \begin{matrix} [(b') : \phi'] ; \ldots ; [(b^{(r)}) ; \phi^{(r)}] ; \\ [(d') : \delta'] ; \ldots ; [(d^{(r)}) ; \delta^{(r)}] ; \end{matrix} (e^{i\theta} \cos\theta)^{h_1/2} \; z_1, z_2, \ldots, z_r \right]$$

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$$= \sum_{R=-\infty}^{\infty} C_R \, e^{-2i\,(R-n)\,\theta} , \qquad \ldots (4.5)$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2} , \; Re\,(n) > 0 .$$

Equation (4.5) is valid, since $f(\theta)$ is continuous and of bounded variation in the open intervals $(-\frac{\pi}{2}$ to $\frac{\pi}{2})$.

Now integrating (4.5) with respect to θ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we get

$$\int_{-\pi/2}^{\pi/2} f(\theta)\, d\theta = \sum_{R=-\infty}^{\infty} C_R \int_{-\pi/2}^{\pi/2} e^{-2i(R-n)\theta}\, d\theta$$

Now using (3.1) and the orthogonality property of exponential functions (2.5), we obtain

$$C_n = \frac{2}{2^{2n}\,\Gamma(n)} \sum_{i,-i} \frac{1}{i}$$

$$H \begin{matrix} 0, \varepsilon : (u'+1, v'+2) ; (u'', v'') ; \quad . ; (u^{(r)}, v^{(r)}) \\ A, C : B'+2, D'+2) ; (B'', D'') ; \ldots ; (B^{(r)}, D^{(r)}) \end{matrix}$$

$$\left[\begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}] : [(b') : \phi'], (1, h_1), (2-2n, h_1/2) ; \\ [(c) : \Psi', \ldots, \Psi^{(r)}] : [(d') : \delta'], (1, h_1), (1-n, h_1/2) ; \end{matrix} \right.$$

$$[(b'') : \phi''] ; \ldots ; [(b^{(r)}) : \phi^{(r)}] ;$$

$$[(d'') : \delta''] ; \ldots ; [(d^{(r)}) : \delta^{(r)}] ;$$

$$\left. \frac{e^{i\pi h_1}}{2^{h_1/2}} \right) \quad z_1, z_2, \ldots, z_r \;] \qquad \ldots (4.6)$$

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With the help of equations (4-5) and (4.6), the formula (4.1) follows. The proofs of the formulas (4.2) to (4.4) can be developed by proceeding on similar lines with the help of (3.2) and (2.6), (3.3) and (2.7), (3.4) and (2.7), respectively.

Acknowledgements

The author is grateful to Prof. H. M. Srivastava (University of Victoria, Canada) for giving suggestions for revising of the paper.

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Jñānābha, Vol. 18, 1988

CERTAIN RESULTS INVOLVING A CLASS OF GENERALIZED LAURICELLA FUNCTIONS

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1. INTRODUCTION

Certain addition and multiplication theorems of Lauricella functions have been discussed by H. Exton [1]. Our aim, in the present note, is to establish some results concerning the generalized Lauricella functions by an application of a fractional differential operator .

2. Notations and Definitions

We consider the following special class of the generalized Lauricella functions due to Srivastava and Daoust ([2, p. 454] ; see also [1, p. 107]) :

$$F\begin{matrix}1:2;\ \ldots\ ;2\\0:2;\ \ldots\ ;2\end{matrix}\left(\begin{matrix}a:b_1,\lambda_1;\ \ldots\ ;b_n,\lambda_n\ ;\\ -:c_1,\mu_1;\ \ldots\ ;c_n,\mu_n\ ;\end{matrix}\ x_1,\ \ldots,\ x_n\right)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}\,(b_1)_{m_1}\,(\lambda_1)_{m_1}\ldots(b_n)_{m_n}\,(\lambda_n)_{m_n}\,x_1^{m_1}\ldots x_n^{m_n}}{(c_1)_{m_1}\,(\mu_1)_{m_1}\ \ldots\ (c_n)_{m_n}\,(\mu_n)_{m_n}\,(1)_{m_1}\ldots(1)_{m_n}}$$

Also, the fractional differential operator D_x^{ε} is defined by (see, e. g., [3, p, 289])

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$$D_x^{\delta}\, x^{a-1} = \frac{\Gamma(a)}{\Gamma(a-\delta)}\, x^{a-\delta-1},$$

where $\delta \neq a$ and $a > 0$.

3. Main Results

In this section, we shall establish the following results :

$$(3.1)\quad \sum_{m_1, \ldots, m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}\,(b_1)_{m_1}\,(\lambda_1)_{m_1} \cdots (b_n)_{m_n}}{(c_1)_{m_1}\,(\lambda_1-\delta_1)_{m_1} \cdots (c_n)_{m_n}\,(\lambda_n-\delta_n)_{m_n}} \cdot \frac{(\lambda_n)_{m_n}\, x_1^{m_1} \cdots x_n^{m_n}}{(1)_{m_1} \cdots (1)_{m_n}}$$

$$\cdot F\left[\begin{matrix} 1:2;\ldots;2 \\ 0:2;\ldots;2 \end{matrix}\right.\left(\begin{matrix} a+m_1+\ldots+m_n : b_1+m_1,\ \mu_1;\ \ldots; \\ - \qquad\qquad\quad : c_1+m_1,\ \mu_1-\nu_1;\ \ldots; \end{matrix}\right.$$

$$\left.\begin{matrix} b_n+m_n,\ \mu_n; \\ c_n+m_n,\ \mu_n-\nu_n; \end{matrix}\ \begin{matrix} y_1, \ldots, y_n \end{matrix}\right)$$

$$= \sum_{p_1, \ldots, p_n=0}^{\infty} \frac{(a)_{p_1+\ldots+p_n}\,(b_1)_{p_1}\,(\mu_1)_{p_1} \cdots (b_n)_{p_n}}{(c_1)_{p_1}\,(\mu_1-\nu_1)_{p_1} \cdots (c_n)_{p_n}\,(\mu_n-\nu_n)_{p_n}} \cdot \frac{(\mu_n)_{p_n}\, y_1^{p_1} \cdots y_n^{p_n}}{(1)_{p_1} \cdots (1)_{p_n}}$$

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$$\cdot F \begin{matrix} 1:2;\ldots;2 \\ 0:2;\ldots;2 \end{matrix} \left(\begin{matrix} a+p_1+\ldots+p_n : b_1+p_1, \lambda_1; \ldots; \\ - \quad : c_1+p_1, \lambda_1-\delta_1; \ldots; \end{matrix} \begin{matrix} b_n+p_n, \lambda_n \ ; \\ c_n+p_n, \lambda_n-\delta_n \ ; \end{matrix} x_1, \ldots, x_n \right).$$

$$(3,2) \quad F \begin{matrix} 1:3;\ldots;3 \\ 0:3;\ldots;3 \end{matrix} \left(\begin{matrix} a : \lambda_1, b_1, \mu_1; \ldots; \lambda_n, b_n, \mu_n \ ; \\ - : \lambda_1-\delta_1, c_1, \mu_1-\nu_1; \ldots; \lambda_n-\delta_n, c_n, \mu_n-\nu_n \ ; \end{matrix} x_1 y_1, \ldots, x_n y_n \right)$$

$$= \sum_{p_1, \ldots, p_n=0}^{\infty} \frac{(a)_{p_1+\ldots+p_n} (b_1)_{p_1} (\lambda_1)_{p_1} \ldots (b_n)_{p_n} (\lambda_n)_{p_n}}{(c_1)_{p_1} (\lambda_1-\delta_1)_{p_1} \ldots (c_n)_{p_n} (\lambda_n-\delta_n)_{p_n}} \frac{(-x_1)^{p_1} \ldots (-x_n)^{p_n}}{(1)_{p_1} \ldots (1)_{p_n}}$$

$$\cdot F \begin{matrix} 1:2;\ .\ ;2 \\ 0:2;\ .\ ;2 \end{matrix} \left(\begin{matrix} a+p_1+\ldots+p_n : b_1+p_1, \lambda_1+p_1; \ldots; \\ - \quad : c_1+p_1, \lambda_1-\delta_1+p_1; \ldots; \end{matrix} \begin{matrix} b_n+p_n, \lambda_n+p_n \ ; \\ c_n+p_n, \lambda_n-\delta_n+p_n \ ; \end{matrix} x_1, \ldots, x_n \right)$$

$$\cdot \prod_{i=1}^{n} {}_2F_1 \left(\begin{matrix} \mu_i, -p_i \ ; \\ \mu_i-\nu_i \ ; \end{matrix} y_i \right)$$

4. Proofs

Proof of (3.1) : The addition theorem for Lauricella function $F_A^{(n)}$, given by H. Exton [1], is

$$(4.1) \quad F_A^{(n)} (a, b_1, \ldots, b_n ; c_1, \ldots, c_n ; x_1+y_1, \ldots, x_n+y_n)$$

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$$= \sum_{p_1, \ldots, p_n = 0}^{\infty} \frac{(a)_{p_1+\ldots+p_n} (b_1)_{p_1} \ldots (b_n)_{p_n} y_1^{p_1} \ldots y_n^{p_n}}{(c_1)_{p_1} \ldots (c_n)_{p_n} (1)_{p_1} \ldots (1)_{p_n}}$$

$$\cdot F_A^{(n)} (a + p_1 + \ldots + p_n, b_1 + p_1, \ldots, b_n + p_n ; c_1 + p_1, \ldots, c_n + p_n; x_1, \ldots, x_n) .$$

Multiplying both sides of (4.1) by $x_1^{\lambda_1 - 1} \ldots x_n^{\lambda_n - 1} y_1^{\mu_1 - 1} \ldots y_n^{\mu_n - 1}$ and applying the operators $D_{x_1}^{\delta_1} \ldots D_{x_n}^{\delta_n} D_{y_1}^{\nu_1} \ldots D_{y_n}^{\nu_n}$ on both sides, we get the result (3.1) after some simplification .

Similarly, results involving Lauricella functions $F_B^{(n)}$, $F_C^{(n)}$, $F_D^{(n)}$ may easily be extended .

Proof of (3.2) : The multiplication theorem for the Lauricella function $F_A^{(n)}$, given by *H*. Exton [1], is

(4.2) $F_A^{(n)} (a, b_1, \ldots, b_n ; c_1, \ldots, c_n ; x_1 y_1, \ldots, x_n y_n) =$

$$= \sum_{p_1, \ldots, p_n = 0}^{\infty} \frac{(a)_{p_1+\ldots+p_n} (b_1)_{p_1} \ldots (b_n)_{p_n} x_1^{p_1} (y_1 - 1)^{p_1} \ldots x_n^{p_n} (y_n - 1)^{p_n}}{(c_1)_{p_1} \ldots (c_n)_{p_n} (1)_{p_1} \ldots (1)_{p_n}}$$

$$\cdot F_A^{(n)} (a + p_1 + \ldots + p_n, b_1 + p_1, \ldots, b_n + p_n ; c_1 + p_1, \ldots c_n + p_n ; x_1, \ldots, x_n)$$

Multiplying both sides of (4.2) by $x_1^{\lambda_1 - 1} y_1^{\mu_1 - 1} \ldots x_n^{\lambda_n - 1} y_n^{\mu_n - 1}$ and applying the operators $D_{x_1}^{\delta_1} D_{y_1}^{\nu_1} \ldots D_{x_n}^{\delta_n} D_{y_n}^{\nu_n}$ on both sides, we get, after some simplification, the result (3.2) .

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Similarly, the known results for Lauricella functions $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$ may be extended .

Acknowledgements

The author is thankful to Dr. R. Y. Denis for his guidance during the preparation of this paper. He is highly indebted to the referee for making certain very useful and valuable suggestions .

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Jñānābha, Vol. 18, 1988

OBTAINING THE HORIZONTAL DEFLECTION OF A FALLING OBJECT WITHOUT THE KNOWLEDGE OF CORIOLIS FORCE

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ABSTRACT

The horizontal deflection relative to the rotating earth is determined of an object dropped from rest from the top of a tower. A salient feature of the present quest is that the relevant equations of motion are formed without bringing in the concept of coriolis forces.

1. INTRODUCTION

The well-known problem of horizontal or eastward deflection of a body dropped from rest from a certain height has been worked out in many text-books of classical dynamics (see [2] and [4]) introducing the idea of coriolis force that arises due to the interaction between the rotation of the Earth and the velocity of the body as observed relative to the Earth . J. M. Potgieter [1] while solving the problem in an alternativ method, first derived the equation of the path of the object with respect to an inertial frame (non-rotating) and then obtained the exact horizontal deflection by transforming the former equation to those with respect to a rotating frame relatively fixed to the earth . Stirling [3] derived the same result almost with the same technique as that of Potgieter [1] but only when the object is allowed to fall in the equatorial plane (zero latitude) .

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Potigieter [1] tackled the problem in unnecessary details incorporating many redundant equations . However, the present design brings forth altogether a different method of attacking the same problem .

2. SOLUTION TO THE PROBLEM

Case 1. Let the particle fall from the top of a tower of height h situated in the equatorial plane and (x,y) be the instantaneous position of the particle with reference to a non-rotating frame XOY, O being the centre of the earth . If g be the acceleration due to gravity, ω the angular velocity of the Earth about the polar axis and a its radius then the equations of motion of the particle are given by

$$\frac{d^2x}{dt^2} = - g \cos \omega t \qquad (1)$$

$$\frac{d^2y}{dt^2} = - g \sin \omega t \qquad (2)$$

along with the initial conditions that at $t = 0$, $x = a + h$, *i. e.* $y=0$,

$$\frac{dx}{dt} = 0, \quad \frac{dy}{dt} = (a + h)\,\omega \text{ due to the earth's rotation} \qquad .. (3)$$

The solutions of equations (1) and (2) with the help of (3) can be obtained as

$$x = a + h - g(1 - \cos \omega t)/\omega^2 \qquad (4)$$

$$y = (a+h)\,\omega t - g(-\sin \omega t)/\omega \qquad (5)$$

which, on expanding cos (ωt) and *sin* (ωt) in series aud neglecting the square and other higher terms of ω, reduce to the form

$$x = a + h - \tfrac{1}{2} gt^2 \qquad \ldots \qquad (6)$$

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$$y = (a + h)\,\omega t - \frac{g\omega t^3}{6} \qquad \ldots \quad (7)$$

If T be the time taken by particle to reach the surface of the earth then at $t=T$, $x = a\cos(wT)$ and $y=d$ (*say*) so that by the *use* of eqns (6) and (7) along with the foregoing approximation we get :

$$h=\tfrac{1}{2}\,gT^2 \qquad (8)$$

$$d=(n+h)\;\omega T - \frac{g\omega T^3}{6} \qquad (9)$$

Eliminating T from (8), (9) and taking into consideration that the foot of the tower in the meantime has already moved eastward a distance equal to a sin $(\omega T) \simeq a\omega T$, the eastward or horizontal deflection, from the foot of the tower, of the object is given by

$$D=d-a\omega T = \frac{1}{3}\;g\omega T^3 = \frac{1}{3}\;2h\omega\left(\frac{2h}{g}\right)^{\frac{1}{2}} \qquad \ldots \quad (10)$$

Case 2. If the particle be dropped from the top of a tower situated in the northern hemisphere at the latitude λ then because of the Earth's rotation about the polar axis, it initially acquires the velocity

$$\vec{V}_0 = \vec{\omega} \times (a+h)\,\hat{r}$$

$$= \vec{k\omega} \times (a+h)\,(i\cos\lambda + k\sin\lambda)$$

$$= j\omega\,(a+h)\cos\lambda \qquad \ldots \quad (11)$$

where the right-handed inertial frame $OXYZ$ consists of the X-axis which is the line of intersection of the equatorial plane with the plane passing through the polaraxis and the radius vector joining the centre

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O of the Eatth to the top of the tower, the Y-axis perpendicular to the former axis and the Z-axis coinncident with the polar axis, (i,j,k) heing the unit vectors associated with the system of axes. Equation (11) reveals that the particle initially gains a velocity equal to $\omega(a+h)$ $cos\ \lambda$ and that too in tne eastward direction indicated by the Y-axis.

If (x,y,z) be the position of the particle at any instnant of time t with respect to this non-rotating frame fixed in space, the equations of motion can be written as

$$\frac{d^2x}{dt^2} = -\frac{qx}{r}, \frac{d^2y}{dt^2} = -\frac{qy}{r}, \frac{d^2z}{dt^2} = -\frac{qz}{r} \quad \ldots (12)$$

$$(r^2 = x^2 + y^2 + z^2)$$

where the initial conditions are

$$t=0,\ x=(a+h)\ cos\ \lambda,\ y=0,\ z=(a+h)\ sin\ \lambda,$$

$$\frac{dx}{dt} = \frac{dz}{dt} = 0 \text{ and } \frac{dy}{dt} = (a+h)\ \omega\ cos\ \lambda \quad \ldots (13)$$

It is a formadible task to obtain solutions to equations (12) subject to the initial conditions. However, approximate closed form solutions can be found out the equations, in the following way

If w is considered to be so small as to be totally negligible, obviously the path of the particle is vertical, *i, e.*, one dimensional accounting for no deflection eastward or otherwise, which is tantamount to solving equations (12) wherein the parameters x, y, z as a way of the first approximation are replaced by their initial values from (13). Because of this fiast approximation, equations (12) reduce to the form

$$\frac{d^2x}{dt^2} = -g\ eos\ \lambda, \frac{d^2y}{dt^2} = 0, \frac{d^2z}{dt^2} = -g\ sin\lambda \quad \ldots (14)$$

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which, as a consequence of (13) and neglecting ω, yleld

$$x = (a+h)\cos\lambda - \tfrac{1}{2} g \cos\lambda\, t^2$$
$$y = 0$$
$$z = (a+h)\sin\lambda - \tfrac{1}{2} g \sin\lambda\, t^2 \quad \ldots \quad (15)$$

The second approximate solutions to equations (14), which can be carried out by not neglecting ω in (13), become

$$y = (a+h)(\omega \cos\lambda)\, t \quad (16)$$

where the expressions for x and z remain the same as in (16), In order to obtain the the third approximate solutions to the equations of set (12), we ban paproximate in its second equation y and r by $\omega(a+h)\cos\lambda\, t$ and $(a+h)$ respectively ; in view of the second approximation (16), initial conditions (13), and ω^2 and other powers of ω being negligibly small, then, by virtue of (13), the integrations of (12) lead to

$$y = (a+h)\,\omega(\cos\lambda)\, t - (1/6)\, g\omega(\cos\lambda)\, t^3 \quad \ldots \quad (17)$$
$$d = (a+h)\,\omega(\cos\lambda)\, T - (1/6)\, g\omega(\cos\lambda)\, T^3 \quad \ldots \quad (18)$$
$$h = \tfrac{1}{2} gT^2 \quad \ldots \quad (19)$$

where T is the time taken by the particle to fall on the surface of the earth. Combining equations (18) and (19) on the lines of (8) and (9), the eastward deflection of the particle relative to the foot of the tower can be obtained as

$$D = d - a\omega(\cos\lambda)\, T$$
$$= (1/3)\, g(\omega\cos\lambda)\, T^2$$
$$= (1/3)\, 2h(\cos\lambda)(2h/g)^{1/2} \quad \ldots \quad (20)$$

To continue this approximation process, we need to replace in the second equation of (12) y by its expression from (17) and r by its initial value as mentioned earlier, so as to obtain with the help of equations (13) and (19) the more accurate deflection .

$$D^1 = 1/3\ 2h\omega \left[1 - \frac{h}{20(a+h)} \right] (\cos\lambda) \left(\frac{2h}{g} \right)^{\frac{1}{2}} \quad \ldots \quad (12)$$

The comparision of (20) with (21) reveals that since $h << a$, the former gives the most ideal result . However, from the theoretical point of view, further continuation of this process entails greater accuracy in determining the deflection of the particle ,

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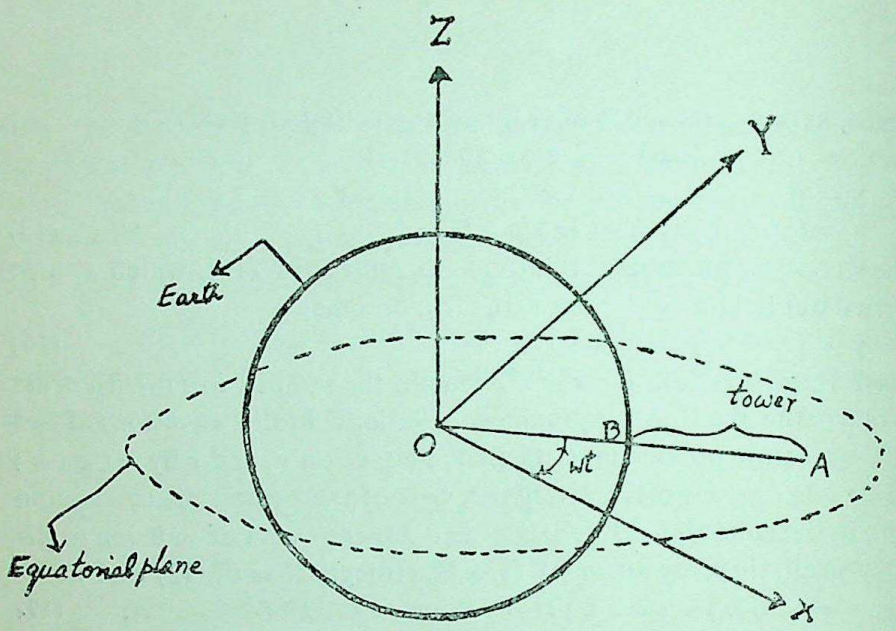

Fig 1. Horizontal deflection in case 1.

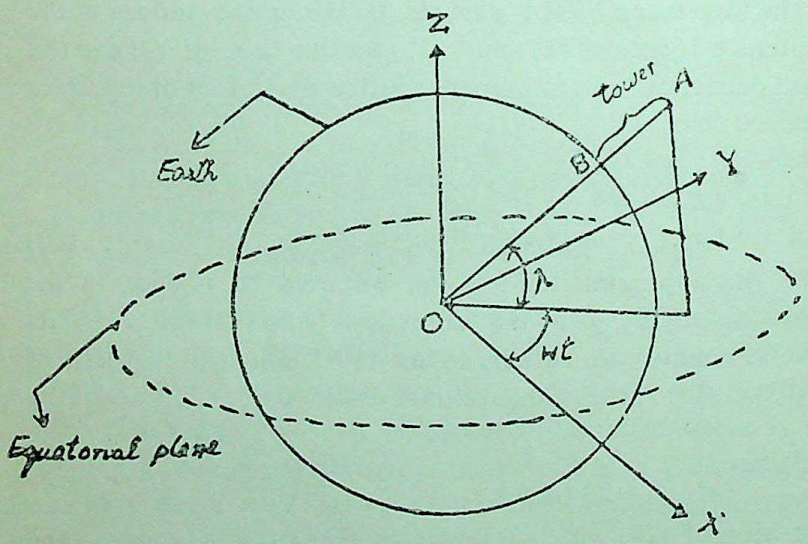

Fig 2. Horizontal deflection in case 2.

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Acknowledgements

I take this opportunity of thanking professor H. M. Srivastava and the learned referee for processing this paper editorially and for their valuable comments .

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Jñānābha, Vol. 18, 1988

ON EXPANSIONS OF q-SERIES OF THREE VARIABLES

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(Received : April 28, 1986 ; Revised : May 5, 1987)

1. INTRODUCTION

Pathan [1] establishd certain expansions of hypergeometric functions of three variables. Recently, Singhal and his co-worker [2] pointed out that all the above expansions are immediate consequence of the simple binomial identity

$$(1.1) \qquad \sum_{r=0}^{n} \binom{n}{r} (a)_{r+k} (c-a)_{n-r} = (a)_k (c+k)_n$$

and used it to establish expansions of certain hypergeometric functions of three variables due to Srivastava [4].

Here we make use of the following basic identity to show how it could be utilized to establish expansion of q-series of three variables, thus, extending the above known expansions to their basic analogues :

$$(1.2) \qquad \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} [a]_{r+k} [c/a]_{n-r} a^{n-r} q^{k(n-r)} = [a]_k [cq^k]_n ,$$

where $$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[q]_n}{[q]_r [q]_{n-r}}$$

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and $[\alpha]_n = (1-\alpha)\ (1-\alpha q)\ \ldots,\ (1-\alpha q^{n-1}),\ [\alpha]_0 = 1$.

The proof of (1.2) is simple, ond we omit the details .

Following Srivastava [5], we define a general q-hypergeometric series of three variables by

$$(1.3)\quad \phi^{(3)}\left[\begin{matrix}(a) : : (b) ; (b') ; (b'') : (c) ; (c') ; (c'') ; x, y, z \\ (e) : : (f) ; (f') ; (f'') : (g) ; (g') ; (g'') ; \lambda, \mu, \nu\end{matrix}\right]$$

$$= \sum_{m, n, p=0}^{\infty} \frac{[(a)]_{m+n+p}\ [(b)]_{m+n}\ [(b')]_{n+p}\ [b'')]_{p+m}\ [(c)]_m\ [(c')]_n}{[(e)]_{m+n+p}\ [(f)]_{m+n}\ [(f')]_{n+p}\ [(f'')]_{p+m}\ [(g)]_m\ [(g)]_n}$$

$$\cdot\ \frac{[(c'')]_p\ x^m y^n z^p q^{\{\lambda m(m+1) + \mu n(n+1) + \nu p(p+1)\}/2}}{[(g'')]_p\ [q]_m\ [q]_n\ [q]_p}$$

where λ, μ and ν are positive to ensure rapid convergence of the series, In case $\lambda=0=\mu=\nu$, $|x|<1$, $|y|<1$ and $|z|<1$.

Also, (a) shall stand for the sequence of A parameters $a_1, a_2, \ldots, a_A$. In what follows, the other symbols will have their usual meaning.

2, EXPANSION FORMULAS

Here we show how (1.2) can be utilized to establish the q-series expansions . To show this, we establish the following expansion

$$(2.1)\quad \sum_{r=0}^{m}\ \sum_{s=0}^{n}\ \sum_{k=0}^{p} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} \begin{bmatrix} p \\ k \end{bmatrix} \frac{[u_r]\ [v/u]_{m-r}\ [u']_{n-s}\ [q/v']_s}{[v]_m\ [qu/v']_n\ [q/u'']_p}$$

$$\cdot\ [q/v'']_k\ [v''/u'']_{p-k}\ u^{m-r}\ (q/v')^{n-s}\ (q/v'')^{p-k}$$

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$$\cdot \phi^{(3)}\left[\begin{matrix}(a) : : (b);\ (b');\ (b'') : uq^{r},\ (c);\ u'q^{n-s},\ (c');\ u''q^{-p},\ (c'');\ xq^{m-r}, \\ (e) : : (f);\ (f');\ (f'') : vq^{m},\ (g);\ v'q^{-s},\ (q');\ v''q^{-k};\ \lambda,\mu,\nu\end{matrix}\ ,\ yq^{n}, z\right]$$

$$= \phi^{(3)}\left[\begin{matrix}(a) : : (b);\ (b');\ (b'') : u,\ (c);\ u',\ (c');\ n'',\ (c'');\ x, y, z \\ (e) : : (f);\ (f');\ (f'') : v,\ (g');\ v'',\ (g'');\ \lambda,\mu,\nu\end{matrix}\right].$$

To prove (2.1), we replace the $\phi^{(3)}$-series by its triple series given by (1.3), interchange the order of summations (valid under the given conditions), and evaluate the inner sums with the help of (1.2), and we get the right side of (2.1) after some simplification .

3. SPECIAL CASES

If we let $q \to 1$ in (2.1) after making some parametaic changes, we get a known expansion due to Singhal and Soni [1, *p*. 38 (1.3)] . Thus, we can also establish the basic analogues of the other expansions due to Singhal and Soni [2]. These would, in turn, include the basic analogues of the expansions given earlier by Pathan [1] .

Acknowledgements

I am thankful to the referee for his valuable comments for the improvement of this paper .

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Jñānābha, Vol. 18, 1988

ON A CHAIN INVOLVING THE *H*-FUNCTION TRANSFORM

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ABSTACRT

In this paper we establish a new chain interconnecting a number of *H*-function transforms having different parameters. Full care has been taken to give an account of all the convergence and existance conditions for the validity of the chain. Since the *H*-function is very general in nature, the corresponding chains for a number of other integral transforms can easily be obtained as special cases of our chain. Similar chains for various simpler integral transforms have been obtained earlier by P, N. Rathie [7], U. C. Jain [6], S. P. Goyal [2], K. C, Gupta and P. K. Mittal [5] ,

1. INTRODUCTION

Throughout this work we use the notation

$$\phi(s) = H\left[f(t) : s : \begin{matrix} m, n & (a_j, \alpha_j)_{1,v} \\ : & : \\ p, q & (b_j, \beta_j)_{1,q} \end{matrix}\right]$$

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$$= \int_0^\infty H_{p,\,q}^{m,\,n}\left[st \middle| \begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right] f(t)\, dt \qquad \ldots (1.1)$$

for the H-function transform defined by Gupta and Mittal [4.*p*. 142]. The Kernel in (1.1) is the well-known H-function which is quite general in nature and includes most of the commonly used funetions as its special cases. The definition of this H-function in terms of a Mellin Barnes contour integral, various conditions satisfied by its paramcters for the convergence of the defining integral, some of its special cases, important properties, and asymptoic expansions, can be referred to in a standard work on the subject by Srivastava, Gupta, and Goyal [8, pp. 10-14]. It has been assumed that the various H-functions occurring in this paper satisfy the conditions corresponding appropriately to those given by (2 2.11(on page 13 in the book referred to above. On account of the general nature of the kernel, this transform provides us with interesting unification and generalizations of a numer of integral transforms (see [8, Chapter 4] for details).

2. MAIN THEOREM

THEOREM. *If*

$$\phi_1(s) = H\left[f(t) : s : \begin{matrix} m_1, n_1 \\ p_1, q_1 \end{matrix} : \begin{matrix} (a_j^{(1)}, \alpha_j^{(1)})_{1,p_1} \\ (b_j^{(1)}, \beta_j^{(1)})_{1,q_1} \end{matrix} \right], \qquad \ldots (2.1)$$

and

$$\phi_k(s) = H\left[t^{c_k - 1}\, \phi_{k-1}\left(\frac{1}{t}\right) : s : \begin{matrix} m_k, n_k \\ p_k, q_k \end{matrix} : \begin{matrix} (a_j^{(k)}, \alpha_j^{(k)})_{1,p_k} \\ (b_j^{(k)}, \beta_j^{(k)})_{1,q_k} \end{matrix} \right] \ldots (2.2)$$

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$$k = 2,3, \ldots, N),$$

then

$$\phi_N(s) = s^{-\sum_{i=1}^{N-1} c_i - (N-1)} \int_0^\infty H^{\sum_{i=1}^{N} m_1, \sum_{i=1}^{N} n_i}_{\sum_{i=1}^{N} p_1, \sum_{i=1}^{N} q_i} \left[st \left| \begin{matrix} I_N \\ \\ J_N \end{matrix} \right. \right] f(t)\, dt \quad \ldots (2.3)$$

where N *is a positive integer greater than* 1,

$$I_N = \left[\left(a_j^{(M)} + \left\{ \sum_{i=1}^{M-1} c_i + (M-1) \right\} \alpha_j^{(M)}, \alpha_j^{(M)} \right)_{1, n_M} \right]_{M=1}^{N-1},$$

$$\left(a_j^{(N)} + \left\{ \sum_{i=1}^{N-1} c_i + (N-1) \right\} \alpha_j^{(N)}, \alpha_j^{(N)} \right)_{1, p_N},$$

$$\left[\left(a_j^{(M)} + \left\{ \sum_{i=1}^{M-1} c_i + (M-1) \right\} \alpha_j^{(M)}, \alpha_j^{(M)} \right)_{n_M+1, p_M} \right]_{M=1}^{N-1}$$

and

$$J_N = \left[\left(b_j^{(M)} + \left\{ \sum_{i=1}^{M-1} c_i + (M-1) \right\} \beta_j^{(M)}, \beta_j^{(M)} \right)_{1, m_M} \right]_{M=1}^{N-1}.$$

$$\left(b_j^{(N)} + \left\{ \sum_{i=1}^{N-1} c_i + (N-1) \right\} \beta_j^{(N)}, \beta_j^{(N)} \right)_{1, q_N},$$

$$\left[\left(b_j^{(M)} + \left\{ \sum_{i=1}^{M-1} c_i + (M-1) \right\} \beta_j^{(M)}, \beta_j^{(M)} \right)_{m_M+1, q_M} \right]_{M=1}^{N-1},$$

it being assumed that the integrals involved in equations (2.1) *and* (2.2) *are absolutely convergent and the following sets of conditions are satisfied*:

$$A_N > 0, \; A'_N > 0, \; |\arg(s)| < \tfrac{1}{2} A'_N \pi,$$

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$$Re\left[\sum_{i=1}^{N-1} c_i + (N-1)\right] - \sum_{M=1}^{N-1} \max_{1\leqslant j\leqslant n_M}$$

$$\left[Re \frac{a_j^{(M)} + \left\{\sum_{i=1}^{M-1} c_i + (M-1)\right\} \alpha_j^{(M)} - 1}{\alpha_j^{(M)}} \right] +$$

$$+ \min_{1\leqslant j\leqslant m_N} Re \frac{b_j^{(N)}}{\beta_j^{(N)}} > 0,$$

$$Re\left[\sum_{i=1}^{N-1} c_i + (N-1)\right] - \sum_{M=1}^{N-1} \min_{1\leqslant i\leqslant m_N}$$

$$\left[Re \frac{b_j^{(M)} + \left\{\sum_{i=1}^{M-1} c_i + (M-1)\right\} \beta_j^{(M)}}{\beta_j^{(M)}} \right] +$$

$$+ \max_{1\leqslant j\leqslant n_N} Re \frac{a_j^{(N)} - 1}{\alpha_j^{(N)}} < 0,$$

where

$$A_N = \sum_{M=1}^{N-1} \left[\sum_{j=1}^{n_M} \alpha_j^{(M)} - \sum_{j=n_M+1}^{p_M} \alpha_j^{(M)} + \sum_{j=1}^{m_{(M)}} \beta_j^{(M)} - \sum_{j=m_M+1}^{q_M} \beta_j^{(M)} \right],$$

$$A'_N = \sum_{j=1}^{n_N} \alpha_j^{(N)} - \sum_{j=n_N+1}^{p_N} \alpha_j^{(N)} + \sum_{j=1}^{m_N} \beta_j^{(N)} - \sum_{j=m_N+1}^{q_N} \beta_j^{(N)}$$

and the notation $\left[(a_j^{(M)}, \alpha_j^{(M)})_1, u_M \right]_{M=1}^{N}$ *stands for a sequence of*

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$\sum_{i=1}^{N} u_i$ *pairs* $(a_j^{(1)}, \alpha_j^{(1)})_{1,u_1}, (a_j^{(2)}, \alpha_j^{(2)})_{1,u_2}, \ldots,$

$(a_j^{(N)}, \alpha_j^{(N)})_{1,u_n}$.

Proof. Interpreting (2.2) for $k=2$ with the help of (1.1), we get

$$\phi_2(s) = \int_0^\infty x^{c_1} H_{p_2, q_2}^{m_2, n_2}\left[sx \,\middle|\, \begin{matrix}(a_j^{(2)}, \alpha_j^{(2)})_{1,p_2} \\ (b_j^{(2)}, \beta_j^{(2)})_{1,q_2}\end{matrix}\right] \phi_1\left(\frac{1}{x}\right) dx \quad \ldots(2.4)$$

Substituting the value of $\phi_1(1/x)$ from (2.1) in (2.4), we get

$$\phi_2(s) = \int_0^\infty x^{c_1} H_{p_2, q_2}^{m_2, n_2}\left[sx \,\middle|\, \begin{matrix}(a_j^{(2)}, \alpha_j^{(2)})_{1,p_2} \\ (b_j^{(2)}, \beta_j^{(2)})_{1,q_2}\end{matrix}\right]$$

$$\cdot \left\{ \int_0^\infty H_{p_1, q_1}^{m_1, n_1}\left[tx^{-1} \,\middle|\, \begin{matrix}(a_j^{(1)}, \alpha_j^{(1)})_{1,p_1} \\ (b_j^{(1)}, \beta_j^{(1)})_{1,q_1}\end{matrix}\right] f(t)\, dt \right\} dx \quad \ldots(2.5)$$

On inverting the order of integrations in (2.5) which is easily seen to be permissible under the conditions stated with the theorem we get

$$\phi_2(s) = \int_0^\infty \left\{ \int_0^\infty x^{c_1} H_{p_1, q_1}^{m_1, n_1}\left[tx^{-1} \,\middle|\, \begin{matrix}(a_j^{(1)}, \alpha_j^{(1)})_{1,p_1} \\ (b_j^{(1)}, \beta_j^{(1)})_{1,q_1}\end{matrix}\right] \right.$$

$$\left. \cdot H_{p_2, q_2}^{m_2, n_2}\left[sx \,\middle|\, \begin{matrix}(a_j^{(2)}, \alpha_j^{(2)})_{1,p_2} \\ (b_j^{(2)}, \beta_j^{(2)})_{1,q_2}\end{matrix}\right] dx \right\} f(t)\, dt \quad \ldots(2.6)$$

Evaluating the x-integral involved in (2.6) with the help of modified form of a result [3, p. 601], we get

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$$\phi_2(s) = \int_0^\infty s^{-c_1-1} H^{m_1+m_2,\, n_1+n_2}_{p_1+p_2,\, q_1+q_2}\left[ts \middle| \begin{matrix} I_2 \\ J_2 \end{matrix}\right] f(t)\, dt \qquad \ldots (2.7)$$

where

$$I_2 = [(a_j^{(1)}, \alpha_j^{(1)})_{1,n_1}, (a_j^{(2)} + (c_1+1)\,\alpha_j^{(2)}, \alpha_j^{(2)})_{1,p_2},$$

$$(a_j^{(1)}, \alpha_j^{(1)})_{n_1+1,\, p_1}]$$

and

$$J_2 = [(b_j^{(1)}, \beta_j^{(1)})_{1,m_1}, (b_j^{(2)} + (c_1+1)\,\beta_j^{(2)}\, \beta_j^{(2)})_{1,q_2},$$

$$(b_j^{(1)}, \beta_j^{(1)})_{m_1+1,\, q_1}]$$

and the following set of conditions are assumed to be satisfied :

$$A_2 \geq 0,\ A'_2 \geq 0,\ |\arg(s)| < \tfrac{1}{2} A'_2\, \pi$$

$$Re(c_1+1) - \max_{1\leq j\leq m_1}\left[Re\,\frac{a_j^{(1)}-1}{a_j^{(1)}}\right] + \min_{1\leq j\leq m_2}\left[Re\,\frac{b_j^{(2)}}{\beta_j^{(2)}}\right] > 0,$$

$$Re(c_1+1) - \min_{1\leq j\leq m_1}\left[Re\,\frac{b_j^{(1)}}{\beta_j^{(1)}}\right] + \max_{1\leq j\leq n_2}\left[Re\,\frac{a_j^{(2)}-1}{\alpha_j^{(2)}}\right] < 0,$$

where

$$A_2 = \sum_{j=1}^{n_1} \alpha_j^{(1)} - \sum_{j=n_1+1}^{p_1} \alpha_j^{(1)} + \sum_{j=1}^{m_1} \beta_j^{(1)} - \sum_{j=m_1+1}^{q_1} \beta_j^{(1)}$$

and

$$A'_2 = \sum_{j=1}^{n_2} \alpha_j^{(2)} - \sum_{j=n_2+1}^{p_2} \alpha_j^{(2)} + \sum_{j=1}^{m_2} \beta_j^{(2)} - \sum_{j=m_2+1}^{q_2} \beta_j^{(2)}.$$

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Now, by assuming the result (2.3) to be true for $N=K$ and proceeding in a manner similar to the above, we can easily arrive at the corresponding result for $N=K+1$. Much of the computational work involved in proving this is straightforward, so we omit the details. Also, from (2.1) and (2.7), we find that the result (2.3) is true for $N=2$. Thus by mathematical induction, we can say that the result (2.3) is true for every positive integer N greater than 1.

3. SPECIAL CASES

1· If in the main theorem, we replace m_k by m_k+1, p_k by m_k, q_k by m_k+1, n_k by 0, $a_j^{(k)}$ by $a_j^{(k)}+b_j^{(k)}$ ($j=1,2, \ldots, m_k, k=1,2, \ldots, N$) and take $\alpha_j^{(k)}=\beta_j^{(k)}=1$ ($j=1,2, \ldots, m_k+1$; $k=1,2, \ldots, N$), $b_{m_k+1}^{(k)}=\rho_k$ therein, it reduces to the following result involving the G-function transforms (see [1] and [8, chapter 4]) :

COROLLARY 1. *If*

$$\phi_1(s)=G\left[f(t): s: \begin{matrix} m_1+1, 0 \\ m_1, m_1+1 \end{matrix} : \begin{matrix} (a_j^{(1)}+b_j^{(1)})_{1, m_1} \\ (b_j^{(1)})_{1, m_1}, \rho_1 \end{matrix}\right] \quad \ldots (3.1)$$

and

$$\phi_k(s)=G\left[t^{c_k-1}\phi_{k-1}\left(\frac{1}{t}\right): s: \begin{matrix} m_k+1, 0 \\ m_k, m_k+1 \end{matrix} : \begin{matrix} (a_j^{(k)}, +b_j^{(k)})_{1, m_k} \\ (b_j^{(k)})_{1, m_k, \rho_k} \end{matrix}\right] \ldots (3.2)$$

$$(k=2, 3, \ldots, N),$$

then

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$$g_N(s) = s^{-\sum_{i=1}^{N-1} c_i-(N-1)} \int_0^\infty G^{\sum_{i=1}^{N} m_i+N,0 \quad n_i}_{\sum_{i=1}^{N} m_i,\ \sum_{i=1}^{N} m_i+N} \left[st \;\middle|\; \begin{matrix} P_N \\ \\ Q_N \end{matrix} \right] f(t)\, dt \qquad \ldots (3.3)$$

where

$$P_N = \left[\left(a_j^{(M)} + b_j^{(M)} + \sum_{i=1}^{M-1} c_i + M - 1 \right)_1,\ m_M \right]_{M=1}^{N},$$

$$Q_N = \left[\left(b_j^{(M)} + \sum_{i=1}^{M-1} c_i + M-1 \right)_1,\ m_M \right]_{M=1}^{N},$$

$$\left[\left(\rho_M + \sum_{i=1}^{M-1} c_i + M - 1 \right) \right]_{M=1}^{N}$$

and the conditions of validity directly obtainable from the main theorem are assumed to be satisfied .

II. If, in Corollary 1, we take $m_k = \rho_k = 0$ ($k = 1, 2, \ldots, N$), it reduces to the following result involving the well-known Laplace transforms :

COROLLARY 2. *If*

$$\phi_1(s) = L[f(t) ; s] \qquad \ldots (3.4)$$

and

$$\phi_k(s) = L\left[t^{c_k-1}\ \phi_{k-1}\left(\frac{1}{t}\right) ; s \right], \ (k = 2, 3, \ldots, N), \qquad \ldots (3.5)$$

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then

$$\phi_N(s) = s^{-\sum_{i=1}^{N-1} c_i - (N-1)} \int_0^\infty H^{N,0}_{0,N}\left[st \,\middle|\, {-\atop R_N}\right] f(t)\,dt \qquad \ldots, (3.6)$$

where $R_N = \left(\sum_{i=1}^{M-1} c_i + M - 1\right)_{M=1}^{N}$ *and the conditions of validity directly obtainable from the main theorem are assumed to be satisfied,*

Acknowledgements

The authors are thankful to Prof. R. P. Agarwal (Vice-Chancellor, University of Rajasthan, Jaipur) and Professor H. M. Srivastava (University of Victoria, Canada) for their constant encouragement . The first author is also thankful to the University Grants Commission of India for providing some financial assistance .

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Jñānābha, Vol. 18, 1988

APPLICATION OF GENERALISED BESSEL'S PARTIAL DIFFERENTIAL OPERATORS TO PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

In Section 1 we have derived some operational formulas with the help of the partial differential operators of generalised Bessel polynomials which may be regarded as the generators of Lie algebra . In section 3 some applications of operational formulas are also pointed out .

1. INTRODUCTION

Isaac I. H. Chen and T. W. Barrett [1] have used Bessel's ordinary differential operators (which raise and lower the index of Bessel's function of the first kind) to solve some second-order linear ordinary differential equations , It may be pointed out that the operators used by Chen and Barrett are not proper Lie elements in order to generate a Lie algebra . The object of the present paper is to use the partial differential operators of M. - P. Chen and C. - C. Feng [2] in connection with the generalised Bessel polynomials, which are regarded as generators Lie algebra, in the derivation of some operational results and finally in the solution of those partial differential equations which can be factorized by means of the generators of the Lie algebra for the generalised Bessel polynomials .

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In a recent paper [3] we have explained the reason why it is impossible to obtain generators of Lie algebra for generalised Bessel polynomials with two operators which raise and lower the index of generalised Bessel polynomials by unity .

In fact, from [2] we have

$$(1.1)\qquad \begin{cases} \left[xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + (n-1)\,y \right] \overset{\alpha}{x}_n = (n+\alpha-1)\,\overset{\alpha+1}{x}_n \\ \left[x^2y^{-1}\frac{\partial}{\partial x} - (nx-\beta)\,y^{-1} \right] \overset{\alpha}{x}_n = \beta\,\overset{\alpha-1}{x}_n , \end{cases}$$

where $\overset{\alpha}{x}_n = \overset{\alpha}{y}_n(x)\; y^{\alpha}$, $\overset{\alpha}{y}_n(x)$ is the generalised Bessel polynomial

(*cf.*, *e.g.*, [4, *p.* 75]) .

If we put

$$(1.2)\qquad \begin{cases} R = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + (n-1)\,y \\ L = x^2y^{-1}\frac{\partial}{\partial x} - (nx-\beta)\,y^{-1} \end{cases}$$

then

$$(1.3)\qquad \begin{pmatrix} 0 & R \\ L & 0 \end{pmatrix} \begin{pmatrix} x_n^{\alpha+1} \\ x_n^{\alpha} \end{pmatrix} = \begin{pmatrix} (n+\alpha-1)\; x_n^{\alpha+1} \\ \beta\; x_n^{\alpha} \end{pmatrix}$$

where

$$x_n^{\alpha} = y_n^{\alpha}(x)\, y^{\alpha} .$$

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Also we notice that

$$(1.4)\quad LR\,[\,y_n^{\alpha}(x)\,y^{\alpha}\,] = \beta\,(n+\alpha-1)\,y_n^{\alpha}(x)\,y^{\alpha},$$

which yields the well-known relation

$$(1.5)\quad \left[x^2\frac{\partial^2}{\partial x^2} + xy\,\frac{\partial^2}{\partial x\partial y} + \beta\,\frac{\partial}{\partial x} - ny\,\frac{\partial}{\partial y} - n(n-1)\right] y_n^{\alpha}(x)\,y^{\alpha} = 0$$

Again

$$[\,R\,,L\,] = -\beta$$

where

$$[\,R\,,L\,] = RL - LR$$

Also we have

$$(1.6)\qquad yL = x^2\frac{\partial}{\partial x} - (nx - \beta)$$

$$xy^{-1}R - yL = xy\frac{\partial}{\partial y} - (\beta - x)$$

2. Derivation of Operational Formulas

Consider the partial differential equation

$$(2.1)\quad xy\,\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + (n-1)\,yu = f(x, y)$$

which is equivalent to

$$x\,\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + (n-1)\,u = y^{-1}f(x, y).$$

Hence the corresponding system of ordinary differential equation is

$$(2.2)\qquad \frac{dx}{x} = \frac{dy}{y} = \frac{du}{y^{-1}f(x, y) - (n-1)\,u}$$

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Solving (2.2) we get $xy^{-1} = c_1$ and

$$(2.3)\quad ux^{n-1} = c_1 \int x^{n-3} f(x, c_1^{-1} x)\, dx + c_2$$

$$= \Psi_1(x, c_1) + c_2, \qquad (say).$$

Hence

$$(2.4)\quad \frac{1}{R}\left[f(x, y) \right] = x^{1-n} \Psi_1(x, c_1) \Big|_{c_1 = xy^{-1}} + x^{1-n} \phi_1(xy^{-1}).$$

where Ψ_1 is given in (2.3) anc ϕ_1 is arbitrary.

Corollary 1. If $y^{-1} f(x, y)$ = a function of x only $= P(x)$, (say), then

$$(2\,5)\quad \frac{1}{R}\left[f(x, y) \right] = x^{1-n} \int x^{n-2} P(x)\, dx + x^{1-n} \phi_1(xy^{-1}),$$

Where ϕ_1 is arbitrary.

Corollary 2. If $y^{-1} f(x,y)$ = a function of y only $= Q(y)$ (say), then

$$(2.6)\quad \frac{1}{R}\left[f(x, y) \right] = y^{1-n} \int y^{n-2} Q(y)\, dy + y^{1-n} \phi_1(xy^{-1}),$$

where ϕ_1 is arbitrary.

Next, consider the partial differential equation

$$(2.7)\quad x^2 y^{-1} \frac{\partial u}{\partial x} - (nx - \beta) y^{-1} u = F(x, y),$$

which is equivalent to

$$x^2 \frac{\partial u}{\partial x} - (nx - \beta) u = y F(x, y).$$

Hence the corresponding system of ordinary differential equation is

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$$(2.8)\quad \frac{dx}{x^2} = \frac{dy}{0} = \frac{du}{yF(x,y) + (nx-\beta)u}$$

Solving (2.8), we get $y = c_1$ and

$$(2.9)\quad u(e^{-\beta/x}\, x^{-n}) = c_1 \int \frac{F(x, c_1)\, dx}{x^{n+2}\, e^{\beta/x}} + c_2$$

$$= \Psi_2(x, c_1) + c_2 \quad (say)$$

Hence

$$(2.10)\quad \frac{1}{L} F(x, y) = e^{\beta/x}\, x^n\, \Psi_2(x, c_1)\Big|_{c_1=y} + e^{\beta/x}\, x^n\, \phi_2(y)$$

where Ψ_2 is given in (2.9) and ϕ_2 is arbitrary.

COROLLARY 8. *If* $yF(x,y)$ = a function of x only $= P(x)$, *(say)*, then

$$(2.11)\quad \frac{1}{L} F(x, y) = e^{\beta/x}\, x^n \int \frac{P(x)}{x^{n+2}\, e^{\beta/x}}\, dx + e^{\beta/x}\, x^n\, \phi_2(y)$$

where ϕ_1 is arbitrary.

COROLLARY 4. *If* $yF(x,y)$=a function of y only $= Q(y)$, *(say)*, then there exists no solution of u.

3. Application of the Operational Formulas

Consider the partial differential equation

$$(3.1)\quad x^3 \frac{\partial^2 u}{\partial x^2} + x^2 y \frac{\partial^2 u}{\partial x \partial y} + \beta x \frac{\partial u}{\partial x} - (nx-\beta)\, y \frac{\partial u}{\partial y} - \{ n(nx-\beta) - (nx-2\beta) \}\, u = f(x,y).$$

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Since

$$[\, xy\frac{}{\partial x} + y^2\frac{\partial}{\partial y} + (n-1)\,y\,]\,[\, x^2 y^{-1}\frac{\partial}{\partial x} - (nx-\beta)\,y^{-1}\,]$$

$$= x^3\frac{\partial^2}{\partial x^2} + x^2y\frac{\partial^2}{\partial x\partial y} + \beta x\frac{\partial}{\partial x} - (nx-\beta)\,y\frac{\partial}{\partial y} - \{\, n\{nx-\beta) - (nx-2\beta)\,\}.$$

The above equation (3.1) becomes

$$RLu = f(x, y).$$

It follows from (2.4) that

$$(3\,2)\quad Lu = x^{1-n}\,\Psi_1(x, c_1)\Big|_{c_1 = xy^{-1}} + x^{1-n}\,\phi_1(xy^{-1})$$

$$(3.3)\quad = F(x,y), (say),$$

where Ψ_1 is given in (2.3) and ϕ_1 is arbitrary. Again it follows from (2.10) that

$$(3.4)\quad u = e^{\beta/x}\,x^n\,\Psi_2(x, c_1)\Big|_{c_1=y} + e^{\beta/x}\,x^n\,\phi_2(y)$$

where Ψ_2 is given in (2.9) and ϕ_2 is arbitrary.

On the other hand, if we consider the equation

$$LRu = F(x, y),$$

which is equivalent to

$$(3.5)\quad x^3\frac{\partial^2 u}{\partial x^2} + x^2y\frac{\partial^2 u}{\partial x\partial y} + \beta x\frac{\partial u}{\partial x} - (nx-\beta)\,y\frac{\partial u}{\partial y} - (nx-\beta)(n-1)\,u = F(x, y),$$

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then it follows from (2.10) that

$$(3\,6) \quad Ru = e^{\beta/x}\ x^n\ \Psi_2(x, c_1)\Big|_{c_1=y} + e^{\beta/x}\ x^n\ \phi_2(y)$$

$$(3.7) \quad = f(x, y), \text{ say},$$

where Ψ_2 is given in (2.9) and ϕ_2 is arbitrary. Hence, from (2.4) it follows that

$$(3.8) \quad u = x^{1-n}\ \Psi_1(x, c_1)\Big|_{c_1 = xy^{-1}} + x^{1-n}\ \phi_1(xy^{-1})$$

where Ψ_1 is given in (2.3) and ϕ_1 is arbitrary.

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Jñānābha, Vol. 18, 1988

FIXED POINT THEOREM FOR MULTIVALUED MAPPINGS UNDER THE CARISTI - KIRK TYPE CONDITION

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(Received : January 5, 1987 ; Revised : February 8, 1988)

ABSTRACT

In this paper we prove a fixed point theorem for a pair of multivalued mappings under the Caristi-Kirk type condition. We also give the condition for the existence of such a fixed point.

1. INTRODUCTION

S. B. Nadler [2] proved a fixed point theorem for multivalued contraction mappings. We use the following notations and definitions as given by him in [2] for our work.

Definitions 1. *Let* (X, d) be a metric space, then

(i) $CB(X) = \{C : C$ *is a nonempty* closed and bounded subset of $X\}$

(ii) $N(\varepsilon, A) = \{x \in X \mid d(x, y) < \varepsilon$ for some $y \in A\}$.

$\varepsilon > 0$ and $A \in CB(X)$.

(iii) $H(A,B) = inf\{A \subset N(\varepsilon, B)$ and $B \subset N(\varepsilon, A)\}$,

$\epsilon > 0$ and $A, B \in CB(X)$.

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The function H is a metric for $CB[X)$ called the Hausdorff metric for $CB(X)$.

Let (X, d_1) and (y, d_2) be two metric spaces. A function $F: X \to CB(y)$ is said to be a multivalued contraction mappIng of X into y if

$$H(F((x), F(y)) \leqslant \alpha\, d_1(x, y) \;\forall\; x, y \in x,$$

$$0 \leqslant \alpha < 1$$

A point x is said to be a fixed point of a multivalued mapping F if $x \in F(x)$.

We shall also make use of the following lemmas, which are noted implicitly in Nadler [2]

Lemma I. *If* $A, B \in CB(X)$ *and* $\bar{x} \in A$, *then, for each positive number* η, *there exists a* $y \in B$ *such that*

$$d(\bar{x}, y) \leqslant H(A, B) + \eta.$$

Lemma 2. *Let* $\{A_n\}$ *be a sequence of sets in* $CB(X)$, *and suppose* $\lim H(A_n, Ao) = 0$, *where* $Ao \in CB(X)$. *Then, if* $x_n \in A_n$, $n = 1, 2, 3, \ldots$, *and* $\lim x_n = x_0$, *it follows that* $x_0 \in A_0$

Recently, Caristi ([1], Theorem 2. 1) presented a generalization of the contraction mapping principle which is now known as Caristi-Kirk fixed point theorem. The following proposition is observed by Felix Browder (See Caristi and Kirk [2]) as the modified version of the famous Caristi-Kirk theorem :

Proposition. *Let* (X, d) *be a complets metric space and* $g: x \to X$ *continuous. Suppose that there exists a lower semicontinuous function* $\phi: X \to R^+$ *such that*

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$d(x, gx) \leqslant \phi(x) - \phi(g(x))$, $x \in X$.

Then, for any $x \in X$, *the sequence* $\{g^n(x)\}$ *converges to a fixed point at* g.

We prove

Theorem 1. *Let* (X, d) *be a complete metric space and* F_1, F_2 *be two continuous multivalued mappings on* X. *Let there be two mappings* ϕ *and* Ψ *on* $CB(x) \to [0, \infty)$ *such that*

(1.1) $H(F_1(x), F_2(y)) \leqslant \phi(x) - \phi(F_1(x)) + \Psi(y) - \Psi(F_2 y))$ *for all* $x, y \in X$.

(1.2) $\phi(A) \leqslant \phi(B)$ *and* $\Psi(A) \leqslant \Psi(B)$, A, B *in* $CB(x)$, *and* $A \subset B$.

Then F_1 *and* F_2 *have a common fixed point. Further, if any of the multivalued mappings is one-one, then this common fixed point is a unique common fixed point of the one-one mapping* :

Remark 1. We will use $\phi(x)$ instead of $\phi(\{x\})$ for the singleton set $\{x\}$, $x \in X$.

Proof. *Let* $x_0, y_0 \in X$ and $x_1 \in F_1(x_0)$. By Lemma 1, $\exists$ a $y_1 \in F_2(y_0)$ such that

$$d(x_1, y_1) < H(F_1(x_0), F_2(y_0)) + \tfrac{1}{2}$$

$$\leqslant \phi(x_0) - \phi(F_1(x_0)) + \Psi(y_0) - \Psi(F_1(y_0)) + \tfrac{1}{2} \text{ by (1.1)}$$

Since $x_1 \in F_1(x_0)$. *i. e.*, $\{x_1\} \subset F_1(x_0)$ by (1.2), we have

$\phi(x_1) \leqslant \phi(F_1(x_0))$, Similarly, $\Psi(y_1) \leqslant \phi(F_2(y_0))$

Hence

$$d(x_1, y_1) < \phi(x_0) - \phi(x_1) + \Psi(y_0) - \Psi(y_1) + \tfrac{1}{2}$$

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Since $y_1 \in F_2(y_0)$, $\exists$ an $x_2 \in F_1(x_1)$ such that

$$d(x_2, y_1) < H(F_1(x_1), F_2(y_0)) + \tfrac{1}{2}$$

$$\leqslant \phi(x_1) - \phi(F_1(x_1)) + \Psi(y_0) - \Psi(F_2(y_0)) + \tfrac{1}{2}$$

$$\leqslant \phi(x_1) - \phi(x_2) + \Psi(y_0) - \Psi(y_1) + \tfrac{1}{2}$$

Since $x_2 \in F_1(x_1)$, $\exists$ *a* $y_2 \in F_2(y_1)$ such that

$$d(x_2, y_2) < \phi(x_1) - \phi(x_2) + \Psi(y_1) - \Psi(y_2) + \frac{1}{2^2}$$

and, since $y_2 \in F_2(y_1)$. $\exists$ an $x_3 \in F_1(x_2)$ such that

$$d(x_3, y_3) < \phi(x_2) - \phi(x_3) + \Psi(y_1) - \Psi(y_2) + \frac{1}{2^2}$$

proceeding in this manner, we obtain $\{x_n\}$ and $\{y_n\}$ such that

$$x_n \in F_1(x_{n-1}),\ y_n \in F_2(y_{n-1}) \text{ and}$$

$$d(x_{n+1}, y_n) < \phi(x_n) - \phi(x_{n+1}) + \Psi(y_{n-1}) - \Psi(y_n) + \frac{1}{2^n}$$

So, for any positive integer n, we have

$$(1.3)\quad \sum_{r=1}^{n} d(x_r, y_r) \leqslant \sum_{r=1}^{n} \{\phi(x_{r-1}) - \phi(x_r) + \Psi(y_{r-1}) - \Psi(y_r)\} + \sum_{r=1}^{n} \frac{1}{2^r}$$

$$< \phi(x_0) - \phi(x_n) + \Psi(y_0) - \Psi(y_n) + 1$$

$$\leqslant \phi(x_0) + \Psi(y_0) + 1, \text{ since } \phi(x_n) \geqslant 0,$$

$$\Psi(y_n) \geqslant 0.$$

Similarly,

$$(1.4)\quad \sum_{r=1}^{n} d(x_r, x_{r+1}) \leqslant \sum_{r=1}^{n} [d(x_r, y_r) + d(x_{r+1}, y_r)]$$

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$$< \phi(x_0) + \phi(x_1) + 2\Psi(y_0) + 2, \text{ and}$$

$\sum_{n=1}^{\infty} d(x_n, x_{n+1})$ is convergent. So $d(x_n, x_{n+1}) \to 0$

as $n \to \infty$ Thus $\{x_n\}$ is a Cauchy sequence and converges to $z_1 \in X$. Since F_1 is continuous and $x_n \in F_1(x_{n-1})$ for $n = 1, 2, 3, \ldots$, by Lemma 2, z_1 is a fixed point of F_1.

Now

$$\sum_{r=1}^{n} d(y_r, y_{r+1}) \leqslant \sum_{r=1}^{n} [(d(x_{r+1}, y_r) + d(x_{r+1}, y_{r+1}))]$$

$$< 2\phi(x_1) + \Psi(y_0) + \Psi(y_1) + 2.$$

Therefore, $\{y_n\}$ is a Cauchy sequence and converges to $z_2 \in X$, so that, by Lemma 2, z_2 *is a* fixed point of F_2. By (1.3),

$\sum_{n=1}^{\infty} d(x_n, y_n)$ is convergent and $\to 0$ as $n \to \infty$

Now $d(z_1, z_2) \leqslant d(z_1, x_n) + d(x_n, y_n) + d(y_n, z_2)$.

Taking limit of both sides as $n \to \infty$, we have

$$d(z_1, z_2) = 0 \text{ and } z_1 = z_2.$$

Let F_1 be one-one and w be any fixed point of F_1. Suppose $z_1 = z_2 = z$. Then

$H(F_1(w), F_2(z)) \leqslant \phi(w) - \phi(F_1(w)) + \Psi(z) - \Psi(F_2(z))$.

Since $w \in F_1(w)$ and $z \in F_2(z)$, by (1.2)

$\phi(w) \leqslant \phi(F_1(w))$ and $\Psi(z) \leqslant \Psi(F_2(z))$. Hence

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$H(F_1(w), F_2(z)) = 0$, so that $F_1(w) = F_2(z)$. Since this is true for any fixed point w of F_1, and z is a fixed point of F_1, we have

$$F_1(z) = F_2(z).$$

Thus $F_1(w) = F_1(z)$ and since F_1 is one-one, $w = z$. This completes the proof.

If both F_1 and F_2 are one-one, this common fixed point would be a unique common fixed point of both F_1 and F_2.

Corollary. *Let (X, d) be a complete metric space and $T = \{F_\lambda : \lambda \in \Lambda\}$ be a family of continuous multivalued mappings on X. Suppose that for each $\lambda \in \Lambda$ there exists ϕ_λ on $CB(X) \to [0, \infty)$ satisfying*

(1.5) $\phi_\lambda(A) \leqslant \phi_\lambda(B)$ $\forall$ A, B *in* $CB(X)$ *with* $A \subset B$.

(1.6) *Let $\exists$ a mapping $F_\alpha \in T$ which is one-one and for any other $F_\beta \in T$ $(\beta \neq \alpha)$ we have*

$$H(F_\alpha(x), F_\beta(y)) \leqslant \phi_\alpha(x) - \phi_\alpha(F_\alpha(x)) + \phi_\beta(y) - \phi_\beta(F_\beta(y))$$

for all x, y in X.

Then the mapping of T has a common fixed point which is the unique fixed point of F,

Remark 2. Theorem 1 is a generalization of the above proposition due to Félix Browder. In fact, by putting $F_1 = I_x$, $F_2 = g$, $x = y$ in Theorem 1, we obtain the above proposition.

2. A FURTHER RESULT

Now we present a result giving the condition for the existence of such a fixed point, considering a multivalued mapping.

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Theorem 2. *Let* (X, d) *be a metric space,* F *be a multivalued mapping on* X *and* ϕ *be a mapping on* $CB(X) \to [0, \infty)$ *such that*

(2.1) $\phi(A) \leqslant \phi(B)\ \forall\ A, B$ *in* $CB(X)$ *with* $A \subset B$.

Suppose $z \in X$ *be such that*

(2.2) $H(F(x), z) \leqslant \phi(x) - \phi(F(x))\ \forall\ x$ in X

then z *is a fixed point of* F. *If* F *is one-one, then* z *is the unique fixed point of* F.

Proof. Let $x_0 \in X$, then by Lamma 2, there exists an $x_1 \in F(x_0)$ such that

$$d(x_1, z) < H(F(x_0), z) + \tfrac{1}{2}$$

$$\leqslant \phi(x_0) - \phi(F(x_0)) + \tfrac{1}{2}, \text{ by (2.2)}.$$

$$\leqslant \phi(x_0) - \phi(x_1) + \tfrac{1}{2}, \text{ by (2.1)}$$

Again, there exists an $x_2 \in F(x_1)$ such that

$$d(x_2, z) \leqslant H(F(x_1), z) + \frac{1}{2}$$

$$\leqslant \phi(x_1) - \phi(x_2) + \tfrac{1}{2}$$

proceeding in this manner, we construct a sequence $\{x_n\}$ such that $x_n \in F(x_{n-1})$, and

$$d(x_n, z) \leqslant \phi(x_{n-1}) - \phi(x_n) + \frac{1}{2^n},\ n = 1, 2, 3 \ldots$$

Then

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$$\sum_{r=1}^{n} d(x_r, z) \leqslant \sum_{r=1}^{n} \{\phi(x_{r-1}) - \phi(x_r) + \frac{1}{2^r}\} \leqslant \phi(x_0) + 1$$

Hence the series $\sum_{n=1}^{\infty} d(x_r, z)$ is convergent so that

$d(x_n, z) \to 0$ *as* $n \to \infty$. Hence $x_n \to z$ *as* $n \to \infty$ Since

F is continuous and $x_n \in F(x_{n-1})$, $n = 1, 2, 3 \ldots$, by Lemma 2 . z is a fixed point of F .

Let F be one-one and z be any fixed point of F, Then

$H(F(z_1), z)) \leqslant \phi(z_1) - \phi(F(z)) \leqslant 0$, so that

$F(z_1) = \{z\}$ This being these of any fixed point of F. we have $F(z) = \{z\} = F(z_1)$, F being one- one, $z = z_1$. This completes the proof of Theorem 2 .

Acknowledgements

The author is extremely grateful to Dr. Ashok Ganguly for his Kind help and valuable guidance in the preparation of this paper , He is indebted to prof . H. M. Srivastave also for his valuable suggestions for this paper .

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Jñānābha, Vol. 18, 1988

A STABILITY CRITERION FOR A MORE DENSE STREAMING FLUID OVER A LESS DENSE FLUID IN POROUS MEDIUM

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(Received : August 7, 1987 ; Revised : February 2, 1988)

ABSTRACT

A criterion for the stability of a more dense fluid layer of finite thickness flowiug over a less dense fluid of infinite extent with a relative horizontal velocity in a porous medium has been obtained ,

1. INTRODUCTION

The instability of two uniform fluids ıo relative horizontal motion separated by a horizontal boundary, when the less dense fluid overlies the more dense one, has been treated by Chandrasekhar [1] The instability of streaming fluids in a porous medium has been considered by Sharma and Spanos [2] , In these studies, both the fluids are assumed to be of infinite extent and that the dansity of the upper fluid is assumed to be less than the density of the lower fluid .

When a more dense fluid overlies a less dense one in the absence of streaming, there will always be instability [1] . But when a more

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dense fluid layer of finite thickness (small) is streaming over a less dense fluid of infinite extent in porous medium one may expect stability also . Situations of more dense fluids streaming over the less dense ones in a porous medium are of frequent occurrence (e. g. water over oil) below the surface of the Earth . The stability of such configurations is of much importance to fluid dynamicists and geophysicists .

The present note is an attempt in this direction wherein we discuss the stability of a more dense fluid layer of finite thickness flowing over s less dense fluid of infinite extent with a relative horizontal velocity in porous medium .

2. THE STABILITY CRITERION AND DISCUSSION

The initial stationary state whose stability we wish to examine is that of an incompressible fluid in which there is horizontal streaming in the x-direction with a velocity $U(z)$ through a homogeneous porous medium . *Let* $\delta\rho$, δp, u_i (u, v, w) and δz_s denote the perturbations in density ρ , pressure p, velocity U_i (U (z), 0, 0) and surfaces of separation z_s respectively ; g, μ, k_1. ε and T_s stand respectively for acceleration due to gravtiy, fluid viscosity, medium permeability, medium porosity and pseudo (effective) interfacial tensions T_s at some prescribed levels z_s . Then the linearized perturbation equations are :

$$(1)\quad \frac{\rho}{\varepsilon}\left(\frac{\partial u}{\partial t} + \frac{U}{\varepsilon}\frac{\partial u}{\partial x} + \frac{\omega}{\varepsilon}\frac{dU}{dz}\right) = -\frac{\partial}{\partial x}\delta p - \frac{\mu}{k_1}u ,$$

$$(2)\quad \frac{\rho}{\varepsilon}\left(\frac{\partial v}{\partial t} + \frac{U}{\varepsilon}\frac{\partial v}{\partial x}\right) = -\frac{\partial}{\partial y}\delta p - \frac{\mu}{k_1}v ,$$

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$$(3)\quad \frac{\rho}{\varepsilon}\left(\frac{\partial\omega}{\partial t}+\frac{U}{\varepsilon}\frac{\partial w}{\partial x}\right)=-\frac{\partial}{\partial z}\delta p-g\delta\rho+\Sigma_s T_s\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\delta z_s\right]$$

$$\delta(z-z_s)-\frac{\mu}{k_1}\omega,$$

$$(4)\quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial\omega}{\partial z}=0,$$

$$(5)\quad \left(\varepsilon\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\delta\rho=-\omega D\rho,$$

$$(6)\quad \left(\varepsilon\frac{\partial}{\partial t}+U_s\frac{\partial}{\partial x}\right)\delta z_s=\omega_s,$$

where $\delta(z-z_s)$ denotes Dirac's delta function, $D=d/dz$ and a subscript s distinguishes the value of the quantity at $z=z_s$.

Assuming the perturbation quantities to have x, y and t dependence of the form

$$(7)\quad exp\, i(k_x x+k_y y+nt),$$

where k_x, k_y are horizontal wave numbers. $k^2=k_x^2+k_y^2$ and n is, in general, a complex constant.

Equations (1)-(6) give

$$(8)\quad D\left[\left\{\frac{i\rho}{\varepsilon}\left(n+\frac{k_x U}{\varepsilon}\right)+\frac{\rho\nu}{k_1}\right\}D\omega-ik_x\omega\left(\frac{\rho}{\varepsilon^2}\right)DU\right]$$

$$-k^2\left[\frac{i\rho}{\varepsilon}\left(n+\frac{k_x U}{\varepsilon}\right)+\frac{\rho\nu}{k_1}\right]\omega=igk^2(D\rho)\frac{\omega}{\varepsilon n+k_x U},\quad (z\neq z_s)$$

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subjected to the boundary conditions :

(9) $\dfrac{\omega}{\varepsilon n + k_x U}$ must be continuous at nn interface ,

and

$$(10)\quad \Delta_s \left[\left\{ \frac{i\rho}{\varepsilon} \left(n + \frac{k_x U}{\varepsilon} \right) + \frac{\rho\nu}{k_1} \right\} D\omega \; ik_x \left(\frac{\rho}{\varepsilon^2} \right) (DU)\omega \right]$$

$$= igk^2 \left[\Delta_s (\rho) - \frac{k^2 T_s}{g} \right] \left(\frac{\omega}{\varepsilon n + k_x U} \right)_s . \quad (for\ z = z_s)$$

Consider now that two uniform fluids of densities ρ_1 nnd ρ_2 be separated by a horizontal boundary at $z=0$ and the density ρ_2 of the upper fluid be more than the density ρ_1 of the lower fluid so that, in the absence of streaming, the configuration is unstable and the medium throughout is assumed to be porous . Let the two fluids be streaming with constant velocities U_1 and U_2 and that the upper fluid layer is finite in width whereas the lower one is infinitely extending. Then in each region of constant ρ and U, Eq. (8) reduces to

(11) $(D^2 - k^2)\,\omega = 0$.

Since $w_1 = 0$ as $z \to -\infty$ and $\omega_2 = 0$ at $z = d$ and since $\omega / (\varepsilon n + k_x U)$ must be continuous on the surface $z=0$, the appropriate solutions for the two regions are

(12) $\omega_1 = A_1 (\varepsilon n + k_x U_1)\, e^{+kz}$, $(z<0)$

and

(13) $\omega_2 = A_2 (\varepsilon n + k_x U_2)\, e^{+kz} + B_2 (\varepsilon n + k_x U_2)\, e^{-kz}$, $(z>0)$

whete A_1, A_2 and B_2 are arbitrary constants .

The condition $\omega_2 = 0$ at $z = d$ implies

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(14) $B_2 = - A_2 e^{2kd}$,

and

$$\omega_2 = A_2 (\varepsilon n + k_x U_2) [e^{kz} - e^{(2d-z)k}] .$$

The continuity of $\omega / (\varepsilon u + k_x U)$ on the surface $z=0$ gives

(15) $A_2 = A_1 / (1 - e^{2kd})$,

and

(16) $\omega_2 = A_1 (\varepsilon n + k_x U_2) [e^{kz} - e^{(2d-z)k}] / (1 - e^{2kd})$.

Applying the boundary condition (10) to the solutions (12) and (16), we obtain the dispersion relation

(17) $$\{ \alpha_2 f(kd) - \alpha_1 \} n^2 + [\frac{2k_x}{\varepsilon} \{ \alpha_2 U_2 f(kd) - \alpha_1 U_1 \} - \frac{i\varepsilon}{k_1} \{ \alpha_2 \nu_2 f(kd) - \alpha_1 \nu_1 \}] n$$

$$+ \left[\frac{k_x^2}{\varepsilon_2} \left\{ \alpha_2 U_2^2 f(kd) - \alpha_1 U_1^2 \right\} - \frac{ik_x}{k_1} \left\{ \alpha_2 \nu_2 U_2 f(kd) - \alpha_1 \nu_1 U_1 \right\} \right.$$

$$\left. - gk \left\{ \alpha_2 - \alpha_1 - \frac{k^2 T}{g(\rho_2 + \rho_1)} \right\} \right] = 0 ,$$

(18) where $f(kd) = \frac{1 + e^{2kd}}{1 - e^{2kd}}$.

Equation (17) yields

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$$(19)\quad Re\,(in) = \frac{-a \pm [\,(p^2+Q^2)^{1/4}\ cos\ \{\,\tfrac{1}{2}\,tan^{-1}\,(Q/p)\,\}\,]}{2\,\{\,\alpha_2 f(kd) - \alpha_1\,\}}$$

where

$$(20)\quad a = \frac{\varepsilon}{k_1}\,\{\,\alpha_2\,\nu_2 f(kd) - \alpha_1\,\nu_1\,\}\,,$$

$$P = \frac{\varepsilon^2}{k_1^2}\left\{\alpha_2\,\nu_2 f(kd) - \alpha_1\,\nu_1\right\}^2 - \frac{4k_x^2}{\varepsilon^2}\left\{\alpha_2\,U_2 f(kd) - \alpha_1\,U_1\right\}^2$$

$$+ 4\left\{\alpha_2 f(kd) - \alpha_1\right\}\left[\frac{k_x^2}{\varepsilon^2}\left\{\alpha_2\,U_2^2 f(kd) - \alpha_1\,U_1^2\right\} - gk\left\{\alpha_2 - \alpha_1 - \frac{k^2 T}{g(\rho_1+\rho_2)}\right\}\right],$$

$$Q = \frac{4k_x\alpha_1\alpha_2 f(kd)}{k_1}\,(\nu_1 - \nu_2)\,(U_1 - U_2)\,,$$

and

$$\alpha_{1,2} = \frac{\rho_{1,2}}{\rho_1+\rho_2}\,,\quad \nu_{1,2} = \frac{\mu_{1,2}}{\rho_{1,2}}\,.$$

It is evident from Eq. (19) that, a necessary and sufficient criterion for stability of a more dense fluid layer of finite thickness flowing over a less dense fluid of infinite extent with a relative horizontal velocity in a porous medium is

$$(21)\quad (P^2+Q^2)\,cos^4\,(\tfrac{1}{2}\,tan^{-1}\,Q/P) < a^4\,,$$

where P, Q and are defined in Eqs. (20).

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For the more dense fluid to be also infinitely extending ($f(kd) \to -1$) and for the non - porous medium ($\epsilon \to 1, k_1 \to \infty$), the criterion (21) reduces to that of Chandrasekhar (inequality (40), [1]). Also, for the case of more dense fluid of infinite extent and in the absence of streaming velocities (*i. e.* the case of Rayleigh - Taylor configuration), the criterion for stability (21) reduces to the result that the system is stable for all wave numbers

$k > \left(\frac{g(\rho_2 - \rho_1)}{T} \right)^{1/2}$ meaning thereby that the small wavelength perturbations are stabilized by surface tension .

Schubert and Straus [3] have shown that a more dense fluid phase overlying a less dense phase of the same fluid in a porous medium can be stable provided the permeability in the vicinity of the interface is not too large . This stabilization occurs due to the phase change and low permeability near the interface . In the present paper we have discussed the stability of a more dense fluid layer of finite thickness flowing over a less dense fluid of infinite extent with a relative horizon al velocity in porous medium . A necessary and sufficient criterion for stability has been derived .

Acknowledgements

It is a pleasure to thank Professor T. J. T. Spanos for helpful discussions while the first author was at the Department of Physics, University of Alberta, where the work was initiated . The authors are also thankful to the referee and to Professoz H. M. Srivastave for suggestions .

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Jñānābha, Vol. 18, 1988

RAYLEIGH-TAYLOR INSTABILITY OF A FLUID-PARTICLE MIXTURE IN POROUS MEDIUM

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(Received : September 25, 1987 ; Revised : February 2, 1988)

ABSTRACT

The Rayleigh – Taylor instability of a fluid – particle mixture in porous medium is considered. For the case of two uniform fluids separated by a horizontal boundary, the stable configuration (lighter fluid overlying the heavier one) remains stable but for unstable configuration (heavier fluid overlying the lighter one), the system is unstable and the growth rates may increase or decay with the increase in particle number density. For the case of exponentially varying density, there is instability for the unstable stratification and the growth rates are found to be both increasing or decreasing with the increase in suspended particle number density and medium permeability.

1. INTRODUCTION

The instability derived from the character of the equilibrium of an incompressible heavy fluid of variable density (*i. e.* of a heterogeneous fluid) is termed the Rayleigh-Taylor instability. Mention may

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be made of two important special cases : (a) two fluids of different densities superposed one over the other ; (b) a fluid with a continuous density stratification . Chandrasekhar [1] has given a detailed account of the Rayleigh-Taylor instability in non-porous medium . When a fluid permeates a porous material, the gross effect is represented by by the Darcy's law . As a result of this macroscopic law, the viscous form in the equations of fluid motion is replaced by the resistance term (μ/k_1) v, where μ is the viscosity of the fluid, k_1 is the permeability of the medium and v is the Darcian (filter) velocity of the fluid . Wooding [4] has considered the Rayleigh instability of a thermal boundary layer in flow through a porous medium .

In geophysical situations . more often than not, the fluid is not pure but may instead be permeated with suspended (or dust) particles, Scanlon and Segel [2] have corsidered the effect of suspended particles on the onset of Bénard convection and found that the caitical Rayleigh number was reduced solely because the heat capacity of the pure gas was supplemented by that of the particles , The thermal instability of fluids in a porous medium in the presence of suspended particles has been studied by Sharma and Sharma [3] . The effects of suspended particles and medium permeability were found to destabilize the layer .

Keeping in mind the geophysical situations and motivated by interest in fluid-particle mixtures, a study has been made of the Rayleigh-Taylor instability of a fluid-particle mixture in porous medium in the present note ,

2. Formulation of the Problem and Perturbation Equations

Consider an incompressible fluid-particle layer consisting of a fluid of density ρ , permeated with suspended particles of density mN,

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arranged in horizontal strata in porous medium . The system is acted on by gravity force g (0,0,$-g$) . Let p ρ and v ($u.v,w$) denote respectively the pressure, density and velocity of the pure fluid ; u (1,r,s), m and $N(\bar{x}, t)$ denote the velocity, mass and number density of the particles respectivety . $K = 6\pi\rho\nu\eta$, where η is particle radius, is constant, ε is porosity and $\bar{x} = (x, y, z)$, In the equations of motion for the fluid, the presence of particles adds an extra force term, proportional to the velocity difference between particles and fluid . The equations of motion and continuity for the fluid are

$$\frac{\rho}{\varepsilon}\left[\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{\varepsilon}\left(\mathbf{v}\cdot\nabla\right)\mathbf{v}\right] = -\nabla p + \rho g - \frac{\mu}{k_1}\mathbf{v} + \frac{KN}{\varepsilon}\left(\mathbf{u}-\mathbf{v}\right), \tag{1}$$

$$V\cdot\mathbf{v} = 0, \tag{2}$$

Following a fluid particle in motion, we have

$$\varepsilon\frac{\partial \rho}{\partial t} + \left(\mathbf{v}\cdot\nabla\right)\rho = 0 . \tag{3}$$

Since the force exerted by the fluid on the particles is equal and opposite to that exerted by the particles on the fluid, there must be an extra force term, equal in magnitude but opposite in sign, in the equations of motion for the particles . The buoyancy force on the paticles is neglected, The interparticle reactions are also not considered because we assume that the distances between particles are quite large compared with their diameter . Under the above assumptions, the equations of motion and continuity for the particles are

$$mN\left[\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\varepsilon}\left(\mathbf{u}\cdot\nabla\right)\right] = KN\left(\mathbf{v}-\mathbf{u}\right), \tag{4}$$

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$$\varepsilon \frac{\partial N}{\partial t} + \nabla \cdot \left(N \mathbf{u} \right) = 0 . \tag{5}$$

The initial state of the system is taken to be a quiescent layer (no settling) with a uniform particle distribution N_0 *i. e.* $\mathbf{v} = (0,0,0)$, $\mathbf{u} = (0,0,0)$ and $N = N_0$, is a constant, The character of the equilibrium of this initial static state can be determined, as usual, by supposing that the system is slightly disturbed and then by following its further evolution .

Lel $\delta\rho$, δp, $\mathbf{v}$ (u, v, w) and $\mathbf{u}$ $(1, r, s)$ denote respectively the perturbations in density ρ . pressure p. velocity of fluid and velocity of particles . Then the linearized perturbed forms of eqs . (1) - (5) become

$$\frac{\rho}{\varepsilon} \frac{\partial \mathbf{v}}{\partial t} = - \nabla \delta p + \mathbf{g} \delta \rho - \frac{\rho \nu}{k_1} \mathbf{v} + \frac{KN}{\varepsilon} \left(\mathbf{v} - \mathbf{u} \right), \tag{6}$$

$$\nabla \cdot v = 0 ,$$

$$mN \frac{\partial \mathbf{u}}{\partial t} = KN (\mathbf{v} - \mathbf{u}) , \tag{8}$$

$$\varepsilon \frac{\partial}{\partial t} = - \omega \frac{d\rho}{dz}, \tag{9}$$

wnere ν $(= \mu/\rho)$ stands for kinematic viscosity of the fluid .

Analyzing the disturbances into normal modes, we seek solutions whose dependence on x, y and t is of the form

$$exp (ik_x x + ik_y y + nt) , \tag{10}$$

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where n is a complex constant and k_x , k_y ($k^2 = k_x^2 + k_y^2$) are wave numbers along x and y directions respectively .

Eliminating u between eqs. (6) and (8) and using (10), eqs . (6)-(9) give

$$\left(n' + \frac{\nu}{k_1}\right) \rho u = - ik_x \delta p \,, \tag{11}$$

$$\left(n' + \frac{\nu}{k_1}\right) \rho v = - ik_y \delta p \,, \tag{12}$$

$$\left(n' + \frac{\nu}{k_1}\right) \rho w = - D\delta p - g\delta\rho, \tag{13}$$

$$ik_x u + ik_y v + D\omega = 0 \,, \tag{14}$$

$$\varepsilon n \delta\rho = - \omega D\rho \,, \tag{15}$$

where

$$n' = \frac{n}{\varepsilon}\left(1 + \frac{uNK\rho}{mu + K}\right) \text{ and } D = d/dz \,.$$

Eliminating δp between eqs . (11)-(13) and using eqs . (14) and (15), we obtain

$$n' [D (\rho D\omega) - k^2\rho\omega] + 1/k_1 [D (\rho\nu D\omega) - k^2\rho\nu\omega]$$

$$= - \frac{gk^2}{\varepsilon n} (D\rho) \omega \,. \tag{16}$$

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3. Two uniform fluids separated by a horizontal boundary

Here we consider the case when two superposed fluids of uniform densities ρ_1 and ρ_2 and uniform viscosities μ_1 and μ_2 are separated by a horizontal boundary at $z=0$. The subscripts 1 and 2 distinguish the lower and the upper fluids respectively. Then, in each region of constant ρ and constant μ, eq. (16) reduces to

$$(D^2-k^2)\, w = 0 . \tag{17}$$

The general solution of eq. (17) is

$$\omega = A\, e^{+kz} + B\, e^{-kz} , \tag{18}$$

where A and B are arbitrary constants.

The boundary conditions to be satisfied in the present problem are :

(i) The velocity ω should vanish when $z \to +\infty$ (for the upper fluid) and $z \to -\infty$ (for the lower fluid).

(ii) $\omega(z)$ is continuous at $z=0$.

(iii) The pressure should be continuous across the interface.

Applying the boundary conditions (i) and (ii). we have

$$\omega_1 = A\, e^{+kz} \; (z < 0), \tag{19}$$

$$\omega_2 = A\, e^{-kz} \; (z > 0), \tag{20}$$

the same constant A has been chosen to ensure the continuity of w at $z = 0$.

The continuity of pressure means that

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$$n'\Delta_0\left(\rho D\omega\right)+\frac{1}{k_1}\,\Delta_0\left(\rho\nu D\omega\right)+\frac{gk^2}{n\varepsilon}\,\Delta_0(\rho)\,\omega_0=0\,. \qquad (21)$$

Applying the condition (21) to the solutions (19) and (20), we obtain

$$n^3+\left[\frac{K}{m}+\frac{2KN}{\rho_1+\rho_2}+\frac{\varepsilon}{k_1}\left(\alpha_1\nu_1+\alpha_2\nu_2\right)\right]n^2$$

$$+\left[\frac{K}{m}\,\frac{\varepsilon}{k_2}\left(\alpha_1\nu_1+\alpha_2\nu_2\right)+gk\left(\alpha_1-\alpha_2\right)\right]n+\frac{gkK}{m}\left(\alpha_1-\alpha_2\right)=0\,, \qquad (22)$$

where $\alpha_{1\,2}=\dfrac{\rho_{1,2}}{\rho_1+\rho_2}$ and $\nu_{1\,2}=\dfrac{\mu_{1,2}}{\rho_{1,2}}$.

(a) *Stable Case* $(\rho_1 > \rho_2)$

For potentially stable case, $\alpha_1 > \alpha_2$ and eq. (22) does not involve any change of sign and so does not allow any positive root. Therefore the system is stable.

(b) Unstable Case $(\rho_2 > \rho_1)$

For potentially unstable case, eq. (22) has one positive root and so the system is unstable. Let n_0 denote the positive root of eq. (22). Then eq. (22) yields

$$n_0^3+\left[\frac{K}{m}+\frac{2KN}{\rho_1+\rho_2}+\frac{\varepsilon}{k_1}\left(\alpha_1\nu_1+\alpha_2\nu_2\right)\right]n_0^2$$

$$+\left[\frac{K}{m}\,\frac{\varepsilon}{k_1}\left(\alpha_1\nu_1+\alpha_2\nu_2\right)+gk\left(\alpha_1-\alpha_2\right)\right]n_0+\frac{gkK}{m}\left(\alpha_1-\alpha_2\right)=0\,. \qquad (23)$$

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The presence of suspended particles, thus, does not affect the Rayleigh-Taylor instability in porous medium qualitatively. The deneminator on the right-hand side of eq. (23] may be positive or negative. Thus with the increase in particle number density for the unstable case, the growth rates may increase or decay.

4. The Case of Exponentially Varying Density

Here we consider the density stratification in the fluid of depth d as

$$\rho(z) = \rho_0 e^{\beta z}, \tag{24}$$

where ρ_0 and β are constants. Assume that $[\beta d << 1, i.e.,$ the variation of density at two neighbouring points in the velocity field, which is much less than the average density, has a negligible effect on the inertia of the fluid,

The boundary conditions for the case of two free surfaces are

$$\omega = D^2\omega = 0 \text{ at } z = 0 \text{ and } z = d. \tag{25}$$

The proper solution of eq. (22) satisfying (25) is

$$\omega = W_0 \sin \frac{s\pi z}{d}, \tag{26}$$

where W_0 is a constant and s is an integer.

Substituting (26) in eq. (16) and neglecting the efsect of heterogeneity of the inertia, we get

$$n^3 + \left[\frac{k}{m}\left(1 + \frac{mN}{\beta}\right) + \frac{\varepsilon\nu}{k_1}\right]n^2 + \left[\frac{\varepsilon\nu}{k_1}\frac{K}{m} - \frac{g\beta k^2}{(\frac{s\pi}{d})^2 + k^2}\right]n$$

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$$-\frac{g\beta k^2 K/m}{(\frac{s\pi}{d})^2+k^2} = 0 . \qquad (27)$$

For the stable stratification ($\beta<0$), eq. (27) does not have any positive root implying thereby that the system is stable . For the unstable stratification ($\beta>0$), the constant term in eq. (27) is negative . This meaus that eq. (27) possesses one positive root implying thereby that the system is unstable . Let n_0 denote the positive root of eq (27) . Then

$$n_0^3 + \left[\frac{K}{m}\left(1 + \frac{mN}{\rho}\right)+\frac{\varepsilon\nu}{k_1}\right] n_0^2 + \left[\frac{\varepsilon\nu}{k_1}\frac{K}{m} - \frac{g\beta k^2}{(\frac{s\pi}{d})^2 + k^2}\right] n_0$$

$$-\frac{g\beta k^2 K/m}{(\frac{s\pi}{d})^2 + k^2} = 0 , \qquad (28)$$

To find the role of particle number density and medium permeability concerning the growth rate of unstable modes, we examine the natures of $d n_0/dN$ and dn_0/dk_1 analytically . Equation (28) yields

$$\frac{dn_0}{dN} = \frac{(K/\rho)\, n_0^2}{\frac{g\beta k^2}{(\frac{s\pi}{d})^2+k^2} - \left[3n_0^2 + 2n_0\left\{\frac{K}{m}\left(1+\frac{mN}{\rho}\right)+\frac{\varepsilon\nu}{k_1}\right\} + \frac{\varepsilon\nu}{k_1}\frac{K}{m}\right]} , \qquad (29)$$

and

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$$\frac{dn_0}{dk_1} = \frac{\dfrac{\varepsilon\nu}{k_1^2}\, n_0 \left(n_0 + \dfrac{K}{m} \right)}{\left[3n_0^2 + 2\, n_0 \left\{ \dfrac{K}{m} \left(1 + \dfrac{\tilde{m} N}{\rho} \right) + \dfrac{\nu}{k_1} \right\} + \dfrac{\varepsilon\nu}{k_1} \dfrac{K}{m} \right] - \dfrac{g\, \xi k^2}{\left(\dfrac{s\pi}{d} \right)^2 + k^2}} \qquad (30)$$

The denominator on the right-hand side of eq . (29) may be positive or negative. Thus with the increase in particle number density for ihe unstable stratification, the growth rates may increase or decay. Similarly eq. (30) implies that with the increase in medium permeability, the growth rates may increase or decay for the unstable stratification .

Acknowledgements

The authors are thankful to the referee and to Professor H. M. Srivastava for suggestions which led to a better presentation of the paper .

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Jñānābha, Vol. 18, 1988

ON CERTAIN TRIPLE INTEGRAL RELATIONS

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(Received : June 26, 1987 ; Revised : August 6, 1987)

ABSTRACT

The aim of this paper is to establish three triple integral relations involving elementary functions . A number of triple integrals can be deduced by proper specialization of the unknown functions g and f occurring in these relations . For the sakc of illustration, one of our integral relations is applied to evaluate a general triple integral involving Srivastava and Panda's multivariable H-function .

1. INTRODUCTION

Many authors have worked on the problem of obtaining integral relatious lnvolving higher classes of special functions of one and more variables (see [3, pp. 72-74 ; pp . 156-161] for details) . In this paper we derive three new integral ralations associated with some elementary functions and illustrate how they can be applied to derive triple integrals which may be of interest .

2. INTEGRAL RELATIONS

$$\int_0^\infty \int_0^\infty \int_0^\infty (x^2+y^2)^{-1/2} \exp\left[i (x^2+y^2+z^2) \left(\frac{x^2-y^2}{x^2+y^2} \right) \right]$$

$$\cos\left[2n \tan^{-1} (y/x) \right] f(x^2+y^2+z^2)$$

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$$\cdot\ g\ \{\tan^{-1}\ (\frac{x^2+y^2}{2})\}\ dx\ dy\ dz$$

$$=\frac{\pi i^n}{2}\int_0^\infty\int_0^\infty J_n\ (u^2+v^2)\ f\ (u^2+v^2)\ g\ \{\tan^{-1}\ (\frac{v}{u})\ du\ dv\ ,$$

... (2.1)

where n is any integer, positive or negative, and the functions f and g are so constrained that the various integrals involved in (2.1) exist .

$$\int_0^\infty\int_0^\infty\int_0^\infty (xy)^{\nu+1/2}\ (x^2+y^2)^{-(\nu+1)}\ exp\ \left[i\ (x^2+y^2+z^2)\ (\frac{x^2-y^2}{x^2+y^2})\ cos\Psi\right]$$

$$\cdot\ J_{\nu-1/2}\ [\frac{2xy}{(x^2+y^2)}\ (x^2+y^2+z^2)\ sin\ \Psi\ ]\ C_n^\nu\ (\frac{x^2-y^2}{x^2+y^2})\ f\ (x^2+y^2+z^2)$$

$$g\ [\tan^{-1}\ (\frac{x^2+y^2}{z})\ ]\ dx\ dy\ dz$$

$$=\ 2^{-(\nu+1)}\ \sqrt{\pi}\ i^n\ (sin\ \Psi)^{\nu-1/2}\ \ C_n^\nu\ (cos\ \Psi)$$

$$\cdot\ \int_0^\infty\int_0^\infty (u^2+v^2)^{-1/2}\ J_{\nu+n}\ (u^2+v^2)\ f\ (u^2+v^2)\ g\ [\tan^{-1}\ v/u\ ]\ du\ dv$$

. . (2.2)

provided that $Re\ (\nu) > -1/2$, $n = 0,1.2\ ,\ \ldots\ ,$ and f and g are so constrained that the various integrals involved in (2.2) exist .

$$\int_0^\infty\int_0^\infty\int_0^\infty (xy)^{2\nu}\ (x^2+y^2)^{-(2\nu+1/2)}\ W^{-\nu}\ Z_\nu\ (W)$$

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$$C_n^{\nu}\left(\frac{x^2-y^2}{x^2+y^2}\right) f(x^2+y^2+z^2)\, g\left[\tan^{-1}\left(\frac{x^2+y^2}{z}\right)\right] dx\, dy\, dz$$

$$= \frac{2^{3\nu}\pi\,\Gamma(n+2\nu)\, Z_{\nu+n}(t)\, t^{-\nu}}{n!\,\Gamma(\nu)}$$

$$\cdot \int_0^{\infty}\int_0^{\infty} (u^2+v^2)^{-\nu} \quad J_{\nu+n}(u^2+v^2)\, f(u^2+v^2)\, g\left[\tan^{-1} v/u\right] du\, dv$$

. . (2.3)

valid under the same conditions as those stated for (2.2) above.

In (2.1), (2.2) and (2.3) $J_{\nu}(x)$ is the Bessel function of the first kind, $C_n^{\nu}(x)$ is the Gegenbauer polynomial, and $Z_{\nu}(x)$ stands for any Bessel function of the first, second, or third kind. Also

$$W = \left[(x^2+y^2+z^2)^2 + t^2 - 2t\left(\frac{x^2-y^2}{x^2+y^2}\right)(x^2+y^2+z^2)\right]^{-1/2}$$

... (2.4)

Proof of (2.1): We have [5, p. 409, Ex. 3]

$$J_n(z) = \frac{1}{\pi i^n}\int_0^{\pi} exp(iz\cos\theta)\cos n\theta\, d\theta \qquad \ldots (2.5)$$

In order to derive the integral relation (2.1), we replace z by r^2, θ by 2θ in (2.5), multiply both sides by $rf(r^2)\, g(\phi)\, dr\, d\phi$, and then integrate the resulting equation with respect to r and ϕ over the intervals $(0,\infty)$ and $(0,\pi/2)$, respectively. We thus get

$$\frac{\pi i^n}{2}\int_0^{\infty}\int_0^{\pi/2} \{J_n(r^2)\, rf(r^2)\, g(\phi)\, dr\}\, d\phi$$

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$$= \int_0^{\infty} \int_0^{\pi/2} \int_0^{\pi/2} exp\,(ir^2 \cos 2\theta)\, \cos 2n\theta\, \frac{r \sin \phi}{r \sin \phi}\, rf(r^2)\, g(\phi)\, dr\, d\phi\, d\theta \qquad \ldots (2.6)$$

If we make the substitutions $x = r \sin \phi \cos \theta$, $y = r \sin \phi \sin \theta$, and $z = r \cos \phi$ on right-hand side of (2.6), and set $u = r \cos \phi$, $v = r \sin \phi$ in left hand side, we are easily led to the integral relation (2.1).

To prove the integral relations (2.3) and (2.4), we start with the known integrals [1, p. 57, Eq. (12) ; p. 48, Eq. (14)] and proceed on the lines similar to those mentioned in the proof of (2.1) .

3. USEFUL DEDUCTIONS

The function g appearing in our integrals (2.1) (2.2) and (2.3) may be chosen appropriately to derive various triple integrals .

For example, if in (2.1), we get

$$g(t) = \cos 2\mu t\, (\sin t)^{\nu} \qquad \ldots (3.1)$$

and simplify the right-hand side of the resulting equation by means of a known integral relation [3, p. 72, Eq. (5. 4. 3)], we arrive at the following result :

$$\int_0^{\infty} \int_0^{\infty} \int_0^{\infty} (x^2+y^2)^{(\nu-1)/2}\, (x^2+y^2+z^2)^{-\nu/2}\, exp\,\{i(x^2+y^2+z^2)$$

$$\left(\frac{x^2-y^2}{x^2+y^2}\right)\Big]\, \cos\,[2n(\tan^{-1} y/x)]\, \cos\,[2\mu\{\tan^{-1}\,(\frac{x^2+y^2}{z}\}]$$

$$f(x^2+y^2+z^2)\, dx\, dy\, dz$$

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$$= \frac{\sqrt{\pi}\, i^n\, \Gamma(\tfrac{1}{2} \pm \mu)\, \Gamma(\frac{1+\nu}{2})\, \Gamma(1+\nu/2)}{8\, \Gamma(1+\nu/2 \pm \mu)} \int_0^\infty f(t)\, J_n(t)\, dt \qquad \ldots (3.2)$$

Re $(\nu) > 0$, n is an integer, positive or negative, and f is so chosen that the integrals on both sides of (3.2) exist.

Now in (3.2), we set

$$f(t) = t^{\lambda-1} H[z_1 t^{\rho_1}, \ldots, z_r t^{\rho_r}]$$

where

$$H[z_1, \ldots, z_r) \equiv H \begin{matrix} O, O : M_1, N_1 ; \ldots ; M_r, N_r \\ P, Q : P_1, Q_1 ; \ldots ; P_r, Q_r \end{matrix}$$

$$\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\right| \begin{matrix} (a_j ; \alpha_j', \ldots, \alpha_j^{(r)})_{1,P} : \\ (b_j ; \beta_j', \ldots, \beta_j^{(r)})_{1,Q} : \end{matrix}$$

$$\left.\begin{matrix} (c_j', \gamma_j')_{1,P_1} ; \ldots ; (c_j^{(r)}, \gamma_j^{(r)})_{1,P_r} \\ (d_j', \delta_j')_{1,Q_1} ; \quad . ; (d_j^{(r)}, \delta_j^{(r)})_{1,Q_r} \end{matrix}\right]$$

is a special case of the multivariable H-function due to Srivastava and Panda [4] (see also [3, p. 251, Eq. (C. 1)]).

Evaluating the resulting integral with the help of a known integral [2, p. 272, Eq. (3. 6)], we arrive at the following interesting triple integral which is believed to be new :

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$$\int_0^\infty \int_0^\infty \int_0^\infty (x^2+y^2+z^2)^{\lambda-(\nu+2)/2} (x^2+y^2)^{(\nu-1)/2} \exp[i(x^2+y^2+z^2)$$

$$\left(\frac{x^2-y^2}{x^2+y^2}\right)] \cos[2n(\tan^{-1} y/x)] \cos\{2\mu(tnn^{-1} \frac{x^2+y^2}{z})\}$$

$$H[Z_1(x^2+y^2+z^2)^{\rho_1}, \ldots, Z_r(x^2+y^2+z^2)^{\rho_r}] dx\, dy\, dz$$

$$= \frac{2^{\lambda-4} \sqrt{\pi}\, i^n \Gamma(\frac{1}{2} \pm \mu \Gamma(\frac{1+\nu}{2}) \Gamma(1+\frac{\nu}{2})}{\Gamma(+\frac{\nu}{2} \pm \mu)}$$

$$H^{O, 1 : M_1, N_1 ; \ldots ; M_r, N_r}_{P+2, Q : P_1, Q_1 ; \ldots ; P_r, Q_r} \left[\begin{matrix} Z_1 2^{\rho_1} \\ \vdots \\ Z_r 2^{\rho_r} \end{matrix}\right| \left(1 \mp \frac{n}{2} - \frac{\lambda}{2} ; \frac{\rho_1}{2}, \ldots, \frac{\rho_r}{2}\right)$$

$$\left.\begin{matrix} (a_j ; \alpha_j', \ldots, \alpha_j^{(r)})_{1,P} ; (c_j', \gamma_j')_{1,P_1} ; \ldots ; (c_j^{(r)}, \gamma_j^{(r)})_{1,P_r} \\ (b_j ; \beta_j', \ldots, \beta_j^{(r)})_{1,Q} ; d_j', \delta_j')_{1,Q_1} ; \ldots ; (d_j^{(r)}, \delta_j^{(r)})_{1,Q_r} \end{matrix}\right] \quad \ldots (3\,3)$$

provided that $Re(\nu) > 0$, $\rho_i > 0$ $(i = 1, \ldots, r)$, n is an integer, positive or negative,

$$Re(\lambda) + \sum_{i=1}^{r} [\rho_i \min_{1 \leqslant j \leqslant m_i} Re(d_j^{(i)}/\delta_j^{(i)})] + n > 0,$$

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$$Re\,(\lambda) + \sum_{i=1}^{r} \left[\rho_i \max_{1 \leqslant j \leqslant n_i} Re\left(\frac{c_j^{(i)} - 1}{\gamma_j^{(i)}} \right) \right] - \frac{3}{2} < 0$$

$$\Lambda_i = - \sum_{j=1}^{P} \alpha_j^{(i)} - \sum_{j=1}^{Q} \beta_j^{(i)} + \sum_{j=1}^{N_i} \gamma_j^{(i)} - \sum_{j=n_i+1}^{P_i} \gamma_j^{(i)}$$

$$+ \sum_{j=1}^{M_i} \delta_j^{(i)} - \sum_{j=m_i+1}^{Q_i} \delta_j^{(i)} > 0$$

and

$$| arg\,(Z_i) | < \tfrac{1}{2} \Lambda_i \pi, \; (i = 1, \ldots, r).$$

The triple integral (3.3) is quite general in character due to general nature of the multivariable H-function involved therein. Thus, by appropriately reducing this multivariable H-function in terms of simpler special functions, one can easily obtain a considerably large number of triple integrals of interest to mathematical analysts and applied mathematicians. The details are omitted.

Acknowledgements

The author is grateful to Dr. S. P. Goyal (University of Rajasthan, Jaipur) for his able guidance and help during the preparation of this paper. He is also thankful to Prof. H. M. Srivastava (university of Victoria, Canada) for making certain useful suggestions and encouragement.

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Jñānābha, Vol. 18, 1988

GENERALIZED FIXED POINT THEOREMS IN METRIC SPACE

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(Received : December 18, 1987)

ABSTRACT

B. E. Rhoades proved several fixed point theorems for mapping satisfying much more general contractive type conditions for three points than those of pittnauer (1975) . They have taken only single operator . In this paper we proved fixed point theorems involving three and four points and at the same time two and more operators . These theorems are more general than those of pittnauer and Rhoades

Theorem 1.

Let T_1 and T_2 be two operators mapping on a complete metric space X, with metric d into itself such that $T_1^p T_2^q (x) = T_2^q T_1^p (x)$, for $p, q \geqslant 1$ and $T_1^p T_2^q (x)$ means T_1 operating p times and T_2 operating q times . Further if for a real number h satisfying $0 \leqslant h \leqslant 1$ and for each $x, y, z \in X$,

$$d (T_1 (x), T_2 (y)) \leqslant h \max \{d (x,y), d (x,z), d (y,z) ,$$

$$d (x,T_1 (x)), d (y,T_2 (y)) , d (x, T_2 (y)) ,$$

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$$d(y, T_1(x)),\ [\, d(x, T_1^p\ T_2^q\ (z)) + d(y, T_1^p\ T_2^q\ (z))\,],$$

$$\frac{[\, d(x, T_1^p\ T_2^q\ (z)) + d(T_1(x), T_1^p\ T_2^q\ (z))\,]}{2},$$

$$\frac{[\, d[\, y, T_1^p\ T_2^q\ (z)) + d(T_2(y), T_1^p\ T_2^q\ (z))\,]}{2},$$

$$\frac{[\, d(x, T_1^p\ T_2^q\ (z)) + d(T_2(y), T_1^p\ T_2^q\ (z))\,]}{2},$$

$$\frac{[\, d(y, T_1^p\ T_2^q\ (z)) + d(T_1(x), T_1^p\ T_2^q\ (z))\,]\,\}}{2}, \qquad \ldots \quad (1)$$

Then T_1, T_2 have a unique common fixed point.

Proof.

Let $x_0 \in X$ and consider a sequence $\{x_n\}$ by setting

$$x_1 = T_1(x_0),\ x_2 = T_2(x_1),\ x_3 = T_1(x_2), \quad \ldots$$

For any set A, let $\delta(A)$ denotes the diameter of A.

Assume, $O(x_0, n) = \{x_0, x_1, \ldots, x_n\}$

and $O(x_0, \infty) = \{x_0, x_1, \ldots\}$

Let $K \leqslant i < j < n$, $x = x_i$, $y = x_j$

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where i is an even integer and j is an odd integer,

$z = x_{i-k}$ and $p+q=k$.

Mow,

$$d(x_{i+1}, x_{j+1}) = d(T_1(x_i), T_2(x_j))$$

$$\leqslant h \max \{ d(x_i, x_j), d(x_i, x_{i-k}), d(x_j, x_{i-k}),$$

$$d(x_i, x_{i+1}), d(x_j, x_{j+1}), d(x_i, x_{j+1}), d(x_j, x_{i+1}) \}$$

$$\leqslant h\,\delta[0(x_{i-k}, j+1+k-i)]$$

$$\leqslant h\,\delta[0(x_{i-k}, n)] \ldots \qquad (2)$$

Note tl at $\delta[0(x_{i-k}, 1)] \leqslant \delta[0(x_{1-k}, 2)] \leqslant \ldots$

so that $\delta[0(x_{i-k}, \infty)] = Sup_n \delta[0(x_{i-k}, n)]$.

From (2) it follows that,

$$\delta[0(x_{i-k}, n)] = d(x_c, x_t)$$

for some choice of integer c, t satisfying,

$i-k \leqslant c \leqslant t, c < t \leqslant n$.

Let n be any positive integer. Then there cxists intcgers c and t with

$i-k \leqslant c \leqslant i, c < t \leqslant n$ such that

$$\delta[0(x_{i-k}, n)] = d(x_c, x_t)$$

If $t \leqslant i+1$, then

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$$d(x_c, x_t) \leqslant M = max\{d(x_c, x_u) : i-k \leqslant c < u \leqslant i+1\}$$

If $t > i$, then,

$$d(x_c, x_t) \leqslant d(x_c, x_{t+1}) + d(x_{t+1}, x_t)$$

$$\leqslant M + h\,\delta[(0(x_{t-k}, n)]$$

$$\leqslant M + hd(x_c, x_t) \text{ and } d(x_c, x_t) \leqslant \frac{M}{1-h}$$

Therefore, $\delta[0\, x_{t-k}, \infty)] \leqslant \frac{M}{1-h}$.

Let $m > n > k$, From (2) with

$i+1=n$, $j+1=m$,

$$d(x_m, x_n) \leqslant h\,\delta[0(x_{n-k-1}, m+k-n+1)]$$

Then there exists a choice of integers c_1, t_1 satisfying $0 \leqslant c_1 \leqslant k$, $c_1 < t_1 \leqslant m+k-n+1$ snch that

$$\delta[0(x_{n-k-1}, m+k-n+1)] = d(x_{n-k-1+c_1}, x_{n-k-1+t_1})$$

$$= d[T_1(x_{n-k-2+c_1}), T_2(x_{n-k-2+t_1})]$$

$$\leqslant h\,\delta[0(x_{u-2k-2+c_1}, t+k+1-c_1)]$$

$$\leqslant h\,\delta[0(x_{n-2k-2}, m-n+2k+2).$$

So that,

$$d(x_m, x_n) \leqslant h^2\delta[0(x_{n-2k-2}, m-n+2k+2)]$$

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In general, with r satisfying

$$n = r(k+1) + s,\ 0 \leqslant s < k+1 .$$

$$d(x_m, x_n) \leqslant h^r \delta [\ 0(x_s, m-n+r(k+1))]$$

$$\leqslant h^r \delta [\ 0(x_0, m)\ ]$$

$$\leqslant h^r \frac{M}{1-h} \to 0 \text{ as } r \to \infty .$$

Therefore $\{x_n\}$ is a Cauchy sequence, hence convergent to a limit say x'.

Now for even n, consider $x = x_n$, $y = x'$, $z = x_{n-k}$, $p+q = k$.

Then by (1) we have,

$$d(x_{n+1}, T_2(x')) = d(T_1(x_n), T_2(x'))$$

$$\leqslant h\, mnx\ \{ d(x_n, x'), d(x_n, x_{n-k}),$$

$$d(x', x_{n-k}), d(x_n, x_{n+1}), d(x', T_2(x')),$$

$$d(x_n, T_2(x')); d(x', x_{n+1})\}$$

Taking the limit as $n \to \infty$, we obtain

$$d(x', T_2(x')) \leqslant hd(x', T_2(x')) \Rightarrow T_2(x') = x'$$

i. e., x' is the fixed point of T_2.

Simiarly by taking n odd and

$$x = x', \quad y = x_n, \quad z = x_{n-k}$$

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We get from (1),

$$d(T_1(x'), x_{n+1}) = d(T_1(x'), T_2(x_n))$$

$$\leqslant h\{d(x', x_n)\, d(x', x_{n-k}), d(x_n, x_{n-k}),$$

$$d(x', T_1(x')), d(x_n, x_{n+1}), d(x', x_{n+1}),$$

$$d(x_n, T_1(x))\}.$$

Tnen taking the limit as $n \to \infty$, we obtain

$$d(T_1(x'), x')) \leqslant h\, d(x', T_1(x') \Rightarrow T_1(x') = x',$$

Showing that x' is the fixed point of T_1,

Thus x' is a common fixed point of T_1 and T_2.

Now we shall show that there is no other common fixed point y' of T_1 and T_2. For is y' were such a point then, $T_1(y') = T_2(y') = y', T_1(x') = T_2(x') = x'$.

considering $x = z = x'. y = y'$, we get

$$d(x', y') = d(T_1(x'), T_2(y')) \leqslant hd(x', y')$$

showing that $x' = y'$

Remark.

If we put $T_1 = T_2$ in theorem 1, we get the following theorem of Rhoades:

Let f be a self mapping of a eomplete metric space X which satisfying for each $x, y, z \in X$, $0 \leqslant h \leqslant 1$ and a positive integer k,

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$$d(f(x), f(y)) \leqslant h \max \{ d(x, y), d(x, z), d(y, z), d(x, f(x)),$$

$$d(y, f(y)), d(x, f(y)), d(y, f(x)),$$

$$[d(x, f^k(z)) + d(y, f^k(z))], [d(x, f^k(z)) + d(f(x), f^k(z))],$$

$$\frac{[d(y, f^k(z)) + d(f(y), f^k(z))]}{2}, [d(x, f^k(z)) + d(f(y), f^k(z))],$$

$$\frac{[d(y, f^k(z)) + d(f(x), f^k(z))]}{2}\}.$$

Theorem 2.

Consider for any positive integers k and m, the self mapping T of a complete metric space X satisfies for $x, y, z_1, z_2 \in X$, $x \neq y$,

$$d(T(x), T(y)) < \max \{ d(x, y), d(x, T(x)), d(y, T(y)),$$

$$\frac{[d(x, T(y)) + d(y, T(x))]}{2}, [d(x, T^k(z_1)) + d(y, T^k(z_1))],$$

$$[d(x, T^m(z_2)) + d(y, T^m(z_2))], \frac{[d(y, T^k(z_1)) + d(T(y), T^k(z_1))]}{2},$$

$$\frac{[d(y, T^m(z_2)) + d(T(y), T^m(z_2))]}{2},$$

$$[d(x, T^k(z_1)) + d(T(y), T^k(z_1))],$$

$$[d(x, T^m(z_2)) + d(T(y), T^m(z_2))],$$

$$[d(x, T^k(z_1)) + d(T(x), T^k(z_1))],$$

$$[d(x, T^m(z_2)) + d(T(x), T^m(z_2))] \ldots \quad (1)$$

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If $T^n(x_0)$ has a cluster point p for some $x_0 \in X$ and if T is continuous on cl $0(x_0)$, the closure of the orbit of x_0, then p is the uniqne fixed point of T and $T^n(x_0) \to p$.

Proof. Let $x_0 \in X$ and consider the sequence $\{x_n\}$,

$\{ x_0, x_1 = T(x_0) , \ldots , x_{n+1} = T(x_n) , \ldots \}$.

For each $k,m \leqslant i$, take, $x = x_i$, $y = x_{i-1}$, $z_1 = x_{i-k}$, $z_2 = x_{i-m}$.

Then we have from (1),

$$d(x_{i+1}, x_i) < max \{ d(x_i, x_{i-1}), d(x_i, x_{i+1}), d(x_i, x_{i+1}), d(x_{i-1}, x_i),$$

$$\frac{[d(x_i, x_i) + d(x_{i-1}, x_{i+1})]}{2}, [d(x_i, x_i) + d(x_{i-1}, x_i)],$$

$$[d(x_i, x_i) + d(x_{i-1}, x_i)], \frac{[d(x_{i-1}, x_i) + d(x_i, x_i)]}{2},$$

$$\frac{[d x_{i-1}, x_i) + d(x_i, x_i)]}{2}, [d(x_i, x_i) + d(x_i, x_i)],$$

$$[d(x_i, x_i) + d(x_i, x_i)], [d(x_i, x_i) + d(x_{i+1}, x_i)],$$

$$[d(x_i, x_i) + d(x_{i+1}, x_i)]$$

$$i.e., \; d(x_{i+1}, x_i) \leqslant max \{ d(x_i, x_{i-1}), \frac{(x_{i-1}, x_{i+1})}{2} \} \quad \ldots (2)$$

We have for $x_{i+1} \neq x_i$, since each $i \geqslant k, m$; otherwise $\{ T^n(x_0) \}$ automatic converges to a fixed point .

Suppose the maximum of the right hand side of (2) is

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$$\frac{d(x_{i-1}, x_{i+1})}{2}.$$

Then we have,

$$d(x_{i-1}, x_i < d\frac{(x_{i-1}, x_{i+1})}{2}$$

$$\leqslant [\frac{d(x_{i-1}, x_i) + d(x_i, x_{i+1})}{2}]$$

which implies $d(x_{i-1}, x_i) < d(x_i, x_{i+1})$ ... (3)

Also $d(x_{i+1}, x_i) < \frac{d(x_{i-1}, x_{i+1})}{2}$ which implises

$$d(x_{i+1}, x_i) < \frac{d(x_{i-1}, x_i)}{2} + \frac{d(x_i, x_{i+1})}{2}$$

i e., $d(x_{i+1}, x_i) < \frac{d(x_{i-1}, x_i)}{2} < d(x_{i-1}, x_i)$ ... (4)

Combining (3) and (4) we have the contradiction,

$$d(x_{i+1}, x_i) < d(x_{i+1}, x_i).$$

Therefore from (2), $d(x_{i+1}, x_i) < d(x_i, x_{i-1})$ [4, Lemma 2]

This shows that $T(p) = p$.

Suppose p and w are fixed points of T, $p \neq w$.

Then from (1) we obtain the contradiction, $d(p, w) < d(p, w)$.

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Since $T^n(x_0) \to p$ for some subsequence $\{n_i\}$ and T is continuous on $cl\ O(x_0)$,

$$d(T^{n_i}(x_0), T^{n_i+1}(x_0)) = d(x_{n_i}, x_{n_i+1}) \to d(p, T(p))$$

$$= 0 \Rightarrow T(p) = p .$$

Fix $\varepsilon > 0$. Then there exists an integer j such that $i \geqslant j$ implies

$$d(x_{n_i}, x_{n_i+1}) < \varepsilon \text{ and } d(x_{n_i}, p) < \varepsilon$$

If we choose $x = x_{n-1}$, $y = p$, $z_1 = x_{n-k-1}$ and $z_2 = x_{n-m-1}$,

then for each $n > n_{j+k}$, $n > n_{j+m}$, we have from (1) and (4),

$$d(x_n, p) < max\{d(x_{n-1}, p), d(x_{n-1}, x)\}$$

$$< max\{d(x_{n-2}, p), d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\}$$

$$= max\{d(x_{n-2}, d(x_{n-2}, x_{n-1})\} \text{ from (4)}$$

$$< \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$< max\{d(x_{n_j}, x_{n_j+1}), d(x_{n_j}, p)\} < \varepsilon$$

i. e., $T^n(x_0) \to p$.

Theorem 3.

Let T_1 and T_2 be two mappings on a complete metric space X on itself such that $T_1^p T_2^q = T_2^q T_1^p$; $p, q \geqslant 1$, then for every $x, y, z_1, z_2 \in X$ and $0 < h < 1$,

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$$d(T_1(x), T_2(y)) \leqslant h \max \{ d(x, y), d(x, T_1(x)), d(y, T_2(x)),$$

$$\frac{[d(x, T_2(y)) + d(y, T_1(x))]}{2},$$

$$[d(x, T_1^p T_2^q(z_1)) + d(y, T_1^p T_2^q(z_1))],$$

$$[d(x, T_2^r T_2^q(z_2)) + d(y, T_1^p T_2^l(z_2))],$$

$$[d(x, T_1^p T_2^q(z_1)) + d(T_1(x), T_1^p T_2^q(z_1))],$$

$$[d(x, T_1^p T_2^l(z_2)) + d(T_1(x), T_1^p T_2^l(z_2))],$$

$$[d(y, T_1^p T_1^q(z_1)) + d(T_2(y), T_1^p T_2^q(z_1))],$$

$$[d(y, T_1^p T_2^l(z_2)) + d(T_2(y), T_1^p T_2^q(z_2))],$$

$$[d(x, T_1^p T_2^q(z_1)) + d(T_2(y), T_1^p T_2^q(z_1))],$$

$$[d(x, T_1^p T_2^q(z_2)) + d(T_2(y), T_1^p T_2^q(z_2))] \quad \ldots (1)$$

Then T_1 and T_2 have a unique common fixed point.

Proof.

Let $x_0 \in X$ and constrtuct a sequence $\{x_n\}$ as follows :

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x_0 , $T_1(x_0) = x_1$, $T_2(x_1) = x_2$, $T_1(x_2) = x_3$, $T_2(x_3) = x_4$. and so on .

Consider $x = x_i$ (i - even) ,

$y = x_{i-1}$, $z_1 = x_{i-p-q}$, $z_2 = x_{i-p-q-1}$. Then from (1) we have ,

$$d(T_1(x), T_2(y)) = d(T_1(x_i), T_2(x_{i-1})) = d(x_i, x_{i+1})$$

$$\leqslant h \max \{ d(x_i, x_{i-1}), d(x_i, x_{i+1}), d(x_{i-1}, x_i),$$

$$\frac{[d(x_i, x_i) + d(x_{i-1}, x_{i+1})]}{2}, [d(x_i, x_i) + d(x_{i-1}, x_i)],$$

$$[d(x_i, x_i) + d(x_{i-1}, x_i)], [d(x_i, x_i) + d(x_{i-1}, x_i)],$$

$$[d(x_i, x_i) + d(x_{i-1}, x_i)], [d(x_{i-1}, x_i) + d(x_i, x_i)],$$

$$[d(x_i, x_i) + d(x_i, x_i)], [d(x_i, x_i) + d(x_i, x_i)],$$

$$[d(x_i, x_i) + d(x_i, x_i)] \},$$

$$\leqslant h \max \{ d(x_i, x_{i-1}), d(x_i, x_{i+1}), d\left(\frac{x_{i-1}, x_{i+1}}{2}\right) \}$$

$$< h \max \{ d(x_i, x_{i-1}), \frac{d(x_{i-1}, x_{i+1})}{2} \}.$$

This implies that $d(x_i, x_{i+1}) \leqslant h\, d(x_i, x_{i-1})$

i. e. , $d(T_1(x), T_2(y)) \leqslant h\, d(x, y)$.

This shows that T_1 and T_2 have a common fixed point [6] .

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Statement of ownership and particulars

about the journal

J Ñ Ā N Ā B H A

1.	**Place of publication**	D. V. Postgraduate College Orai–285001, U. P., India
2.	**Periodicity of publication**	Annual
3.	**Printer's name**	*Dr. Anil Kumar Gupta*
	Nationality	Indian
	Address	Navin Printing Press, 98, Gopal Ganj Orai–285001, U. P.
4.	**Publisher's name**	*Dr. R. C. Singh Chandel* For Vijnana Parishad of India
	Nationality	Indian
	Address	D. V. Postgraduate College Orai-285001, U. P., India
5.	**Editor's name**	*Dr. R. C. Singh Chandel*
	Nationality	Indian
	Address	D. V. Postgraduate College Orai–285001, U. P., India
6.	**Name and address of the individuals who own the journal and partners or share hoders holding more than one percent of tde total capital**	Vijnana Parishad of India, Address : D. V. Postgraduate College Orai-285001, U. P., India

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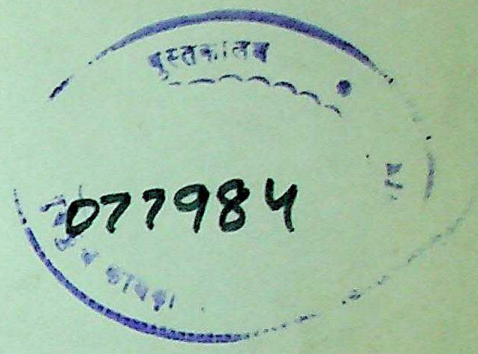

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